Currency Speculation in a Game-Theoretic Model of International Reserves

Carlos J. Pérez* Manuel S. Santos†

This paper is concerned with speculation in currency markets. We consider a game-theoretic setting between a unit mass of speculators and a government that holds international reserves to defend a currency peg. The payoff from a devaluation is allowed to depend on the excess supply of domestic currency. No shape restrictions are imposed on the payoff function, and so speculators' actions may fail to be strategic complements. We extend known methods of analysis to deal with existence and uniqueness of a threshold equilibrium. We also perform various numerical exercises, and offer further insights into the role of heterogeneous information, transaction costs, the optimal level of international reserves, and the widening of currency bands.

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1. Introduction

International currency reserves allow governments to follow exchange rate policies by intervention in the foreign exchange market. In a typical currency peg, international reserves are held to cushion imbalances from foreign trade and capital flows. A currency peg, however, may spur speculation: If a mass of traders considers that the stock of international reserves is too low, then they may flock to short the currency. The stock of reserves may be depleted—and the government will be forced to abandon the peg. A speculative attack can result in a sudden devaluation with severe negative effects on the financial and real sectors because of monetary frictions and real rigidities. A currency crisis may then unfold.

There are numerous examples of currency attacks, and there are long-standing issues regarding the optimal degree of transparency, transaction costs, and the level of international reserves to protect the value of a currency. Some of these issues became apparent in the following three important episodes of currency speculation.

Since its inception in 1979, the European exchange rate mechanism (ERM) experienced constant tensions that translated into a substantial number of currency realignments. After a swing of devaluations affecting some major currencies (e.g., the British pound, French franc, and Italian lira) the ERM essentially collapsed in 1993 as it moved to a much broader currency band. Then, currency values stabilized. The widening of a currency band may thus dissuade speculation, but most reduced-form models of exchange rate determination are not built to quantify these effects.

The 1994 currency crisis of the Mexican peso brought up some transparency issues. Calvo (1998) argued that with uncertainty about the fundamentals, financial crises may spread by contagion and herding behavior. The International Monetary Fund (IMF) has set up the Special Data Dissemination Standards (SDDS) for all member countries. Disclosure practices of foreign currency reserves and other macro variables have varied over time and across member countries, but it is often argued that one must adhere to the highest possible standards of transparency (see op. cit.). Further analytical work in this area should prove valuable to understand the effects of asymmetric information on the optimal amount of international reserves.¹

In the Asian currency crisis that started in 1997, the Thai government spent billions of dollars of its foreign currency reserves to defend its baht against speculative attacks. According to most analysts, lack of timely response by the IMF and other institutions such as the US Fed may have triggered the crisis. For instance, Radelet and Sachs (2000) conclude that policy mistakes at the onset of the crisis by Asian governments and poorly designed international rescue programs led to a full-fledged financial panic.

¹A case in point is the current European sovereign debt crisis. Massive coordination by European countries has proved to lower yield spreads. On May 14, 2010, the European Central Bank (ECB) launched the so-called Securities Market Program to purchase Euro denominated government bonds. ECB interventions are performed in secrecy. Hence, it transpires from these policy practices that noisy information may be most effective to fight speculation. This approach would go against traditional IMF guidelines.
The crisis spread to various parts of the world, and called attention to the role of international reserves and cooperation. Recently, in response to the current global turmoil that started with the subprime financial crisis, the IMF, the G-20, and the European Union have geared towards restrictions to global finance that favor introduction of a Tobin tax. Again, to evaluate these proposals we need to have in hand an analytical framework that can address the role of transaction costs and speculative behavior.

In summary, the aforementioned episodes of currency crisis highlight the role of currency bands, noisy monetary policy, and international reserves and cooperation. More recently, the policy debate has centered on the role of transaction costs. There seems to be a shortage of quantitative work to assess the importance of these factors together with the propagation mechanisms that may enhance a massive speculative attack. Here, we analyze a stylized global game in which the payoff from speculation is allowed to depend on the excess supply of currency. To accommodate further effects of government’s policies on international reserves and endogenous currency depreciation, no shape restrictions are imposed on the payoff function. Then, speculators’ actions are not necessarily strategic complements everywhere. We show existence and uniqueness of equilibrium, and report various quantitative experiments.

The paper is organized as follows. In Section 2 we motivate our approach with some related models of currency speculation, bank runs, and bubbles. Section 3 presents our model of currency speculation with explicit modeling of international reserves and asymmetric information. In this model, the assumption of global strategic complementarity in players’ actions appears to be overly restrictive. (For instance, following Obstfeld (1996), in a successful attack the given stock of reserves may be prorated among those short-selling the currency.) In spite of the lack of global strategic complementarity, in Section 4 we show existence and uniqueness of a threshold equilibrium. We also perform several numerical exercises to evaluate the role of asymmetric information, transaction costs, and the payoff from currency depreciation on the optimal amount of reserves. Section 5 discusses several extensions of our basic analysis. We conclude in Section 6 with a systematic account of our main findings.

2. Strategic Complementarity

The literature on financial and currency speculation is quite broad (Allen and Gale, 2007; Brunnermeier, 2001; Burnside, Eichenbaum, and Christiano, 2008). In this section we review a few models to provide some background and motivation for our future development. Most of our discussion will center on the assumption of strategic complementarity.

2.1. Self-fulfilling equilibria

Currency attacks may be self-fulfilling as a consequence of multiple equilibria. The mere belief of an impending attack may induce speculators to flee the currency. The capital flight is then justified by a devaluation that confirms the initial beliefs. The point has been neatly discussed by Obstfeld (1996), who presents an illustrative example of the so called second-generation models of currency crisis. The model has two pure-strategy equilibria.

Two private holders of domestic currency must decide whether to hold or to sell the currency. Each holder has 6 units of the domestic currency, and will bear a cost equal to one upon selling. The pegged rate is set at par with the international currency. The government owns 10 units of reserves to sustain the peg, yet a 50% devaluation sets off if those reserves are depleted. This is the corresponding payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>hold</th>
<th>sell</th>
</tr>
</thead>
<tbody>
<tr>
<td>hold</td>
<td>0,0</td>
<td>0,−1</td>
</tr>
<tr>
<td>sell</td>
<td>−1,0</td>
<td>3/2,3/2</td>
</tr>
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Figure 1: The intermediate reserve game of Obstfeld (1996).

This game has two pure-strategy equilibria; one in which both holders sell, and another one in which no holder sells. There is strategic complementarity in that selling becomes profitable only if the other holder sells. But the gains from depreciation for one holder are inversely related to the holdings of the other agent since the government has released a fixed quantity of reserves; in other words, in the (sell, sell)-equilibrium an agent would be better off if the other holder owned only 5 units of the domestic currency.

Several models of currency speculation have emphasized the existence of multiple equilibria, as well as the need of coordination devices over those equilibria. Some empirical literature (Kaminsky, Lizondo, and Reinhart,
suggested unusual behavior in the periods preceding a crisis. If a certain aggregate leading indicator surpasses a given threshold value, then it may be perceived as a signal that a currency crisis may occur, and a trigger to the \((sell, sell)\)-equilibrium. Note that if this symmetric game were written in sequential form then the \((sell, sell)\)-equilibrium would be the only one surviving backwards induction. This simple observation seems to be neglected in the so-called second-generation models of currency crises that simply stress multiplicity of equilibria without regard to further properties of these equilibria.

Still, the above game lacks predictive power since the size of transaction costs and the payoff from currency depreciation do not affect equilibrium outcomes for a wide range of parameters. This rather odd result comes from perfect information. Players with diffused information about the stock of reserves may not be so forthcoming about shorting the currency and bear the transaction cost. Each player would have to make conjectures about the beliefs of the other players, who in turn will be making conjectures about the beliefs of the beliefs of all the other players, and so on.

### 2.2. Global games

Morris and Shin (1998) propose a two-stage game between the government and a continuum of speculators. In the first stage, each speculator has to choose whether or not to sell short one unit of the domestic currency at a certain cost \(c > 0\), and in the second stage the government has the choice of defending the peg \(e^*\). If the government defends, the price stays at the original level \(e^*\) and the speculators who attack earn nothing and pay the cost of short-selling. If the government does not defend, the exchange rate falls to \(f(\theta)\), where \(f\) is increasing in the state \(\theta\) of the fundamentals; hence, the speculators exchanging the currency earn the price difference minus the cost: \(e^* - f(\theta) - c\). The government's payoff upon defending is written as:

\[
v - c(\theta, \alpha).
\]

This value increases with the state \(\theta\) of the fundamentals and goes down with the mass \(\alpha\) of speculators attacking the currency.

It should be understood that speculators' actions are strategic complements in a rather strong sense: The chances of a devaluation increase with the mass of speculators who short the currency and the payoff of a devaluation is independent of the underlying level of international reserves and the mass of speculators shorting the currency. That is, all speculators can sell the domestic currency at the pegged price if the government does not defend. This payoff function occurs if the government has access to a large amount of reserves and is committed to honor all desired trades by the speculators.

As in the above intermediate game of Obstfeld (1996), under common knowledge there exist two equilibria for a suitable range of parameter values: One in which no speculator attacks and the government maintains the peg, and another one in which all speculators attack and the government accommodates. Morris and Shin show that this multiplicity of equilibria is not robust: Asymmetric information about the state of the fundamentals gives rise to a unique equilibrium. Their method of proof hinges upon the assumption of global strategic complementarity. In our model below, speculators' actions are not strategic complements everywhere because in the event of a currency depreciation the gains from trading that accrue to each speculator are a function of the mass of speculators shorting the currency (cf. Obstfeld, 1996).

### 2.3. Bank runs

As is well known, Diamond and Dybvig (1983) provide a model of demand-deposit contracts in which there are two equilibria: An efficient equilibrium in which only investors facing liquidity shocks withdraw early, and a bank run equilibrium in which all investors withdraw and the bank vanishes. The early interruption of long-term investments may entail a loss. Hence, as in the case of currency attacks, the fear of an imminent run may propel a large cash withdrawal—vindicating the initial beliefs.

This model is subject to the same criticisms above because of the assumption of perfect information. Goldstein and Pauzner (2005) propose a similar model of bank runs with asymmetric information à la Morris and Shin (1998) and show that the multiplicity of equilibria washes out. In Goldstein and Pauzner (2005) depositors' actions are not strategic complements everywhere. More specifically, conditioning upon the bank failing, as more depositors withdraw their funds, the lower is their share on the bank's liquidation value. There are, however, one-sided strategic complementarities: If

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\(^2\)As in our model below, the domestic currency always depreciates if the government does not defend and so there must be an excess supply of the domestic currency at the pegged price—even if no speculator attacks.
the bank survives then early withdrawals reduce depositors' payoffs. Their proof of uniqueness of equilibrium builds upon this property of the payoffs.

2.4. Bubbles

Our work is also related to the theory of bubbles in behavioral finance. In Abreu and Brunnermeier (2003), a continuum of speculators face a mass of behavioral traders holding a stock with abnormal price growth. The stock price will continue to grow at the bubbly rate as long as there are enough speculators holding the stock; once trading surpasses this threshold, the bubble bursts immediately. Hence, each speculator cares about the optimal timing to sell the stock.

Although the problem is framed in a richer, dynamic setting, the similarities between their problem and ours are evident. A speculator should sell immediately in the belief of a proximate collapse, and wait otherwise; further, the belief of a proximate collapse is self-confirming. These authors, however, did not pursue the global games line, because in their own words (op. cit., p. 177): “In the richer strategy set of our model strategic complementarity is not satisfied and the global games approach does not apply.”

Speculators would like to preempt others because the bubble bursts if the amount of speculators selling the asset outweighs the amount of behavioral traders buying the asset—behavioral traders play the same role as the government in a model of currency crisis. Given the similarities between the two problems, we expect that our work will serve as a first step towards extending global games to dynamic speculation in which the assumption of strategic complementarity may often be quite restrictive.

3. The Model

Our analytical framework for currency speculation considers a simultaneous-move game between a government and a unit mass of speculators. The government wants to defend the peg—absorbing the domestic currency until the stock of international reserves $R$ is depleted. The state of the world is thus represented by the available amount $R$ of international reserves. Government intervention is necessary because the peg differs from the equilibrium rate: There is an excess supply of $s_{\epsilon}$ units of the domestic currency which would call for a devaluation.

We may interpret that $R$ exemplifies the government’s degree of commitment to the exchange rate defense (Obstfeld, 1996). Hence, rather than as an exogenous limit, we can think of quantity $R$ as the outcome of a previous, yet not modeled, deliberation by the government since funds may be drawn from international capital markets or may be saved for other purposes. This information is usually hard to guess by both the government and the traders as it can depend on unexpected external forces.

Each speculator can attack (or not attack) the peg by shorting one unit of the domestic currency at a cost $c > 0$. Let $s$ be the mass of speculators who short the currency. Then, for $s + s_{\epsilon} > R$ the domestic currency depreciates. Otherwise, the peg survives. In the event of a depreciation, the payoff from short selling the currency is determined by a twice continuously differentiable function with bounded derivatives $h(z) : (0,1) \rightarrow \mathbb{R}_+$ over the ratio $0 < z = \frac{R}{s + s_{\epsilon}} < 1$. Therefore, for a speculator shorting one unit of the currency the payoff is $-c$ if the peg survives, and $h(z) - c$ if it does not survive. A speculator not attacking the peg gets zero in any case.

We would like to remark that function $h$ can encompass various trading environments: (i) $h'(z) = 0$ for $0 < z < 1$ means that the payoff from currency depreciation does not depend on the level of reserves or the amount of speculators shorting the currency; this case would occur under a fixed depreciation rate $\delta$ to be applied to the excess supply of currency. And (ii) $h(z) = \delta z$ for $0 < z < 1$ means that the payoff from currency depreciation is inversely related to the amount of speculators shorting the currency; this case would occur under a fixed depreciation rate $\delta$ with a fixed volume of international reserves to be prorated by all speculators short-selling the currency (Obstfeld, 1996).

In summary, currency speculation is modeled in a static game-theoretic framework between a government and a continuum of speculators. The gains from speculation are determined by a nonlinear function $h(z)$ where $0 < z < 1$ is defined as the amount of reserves $R$ over the total supply of units of the domestic currency $s + s_{\epsilon}$. If function $h$ is increasing over the domain $0 < z < 1$, the gains from speculation go down with the mass of speculators attacking the currency. That is, speculators’ actions are strategic substitutes over the domain $0 < z < 1$ in which the peg is abandoned.
3.1. Perfect information

Before getting into further technical issues it may be useful to study the simple case in which the available amount of international reserves $R$ is common knowledge. Let us assume the following condition

$$c < h \left( \frac{R}{1 + s_c} \right).$$

(1)

As in Obstfeld (1996), the following three scenarios may emerge:

- **The low-reserve game**: $R < s_c$. The government does not have enough reserves to defend the peg even if no speculator shorts the currency. Hence, a devaluation will be forthcoming. In the low-reserve game, short-selling the currency is a dominant strategy. Therefore, there is a unique equilibrium in which all speculators attack and the peg is abandoned.

- **The intermediate-reserve game**: $s_c \leq R < 1 + s_c$. The government may or may not have enough reserves to defend the peg: If $R < s + s_c$, the peg will be abandoned. Shorting the currency is the optimal strategy for all speculators who believe that $s + s_c > R$, and not shorting the currency is the optimal strategy for those who believe that $s + s_c \leq R$. Moreover, both beliefs are self-confirming when held equally across the population of speculators. Therefore, in the intermediate-reserve game there are two equilibria in pure strategies: One in which all speculators attack and the peg is abandoned, and another one in which no speculator attacks and the peg survives. There is also an equilibrium in mixed strategies.

- **The high-reserve game**: $1 + s_c \leq R$. The government has enough reserves to defend the peg even if all speculators short the currency. Hence, the peg will survive. In the high-reserve game, shorting the currency becomes a strictly dominated strategy. Therefore, there is a unique equilibrium in which no speculator attacks and the peg survives.

3.2. Imperfectly observed reserves

In our global game, speculators hold certain beliefs about $R$. Assume that each speculator has a uniform prior on the interval $[\bar{R}, \tilde{R}]$. Let $\bar{R} < s_c$ and $\bar{R} > 1 + s_c$. Each speculator receives a conditionally independent signal $x$ which is again uniformly distributed over the interval $[\bar{R} - \varepsilon, \bar{R} + \varepsilon]$ (with constant $0 < \varepsilon < 1/2$). Then, the posterior belief about $R$ of a speculator who receives the signal $x$ is uniform over the interval $[x - \varepsilon, x + \varepsilon]$. Note that parameter $\varepsilon$ is both a measure of the precision of each signal and the degree of informational asymmetry among speculators since signals are conditionally independent. Varying the degree of dependence between the signals would allow us to disentangle both features.

It should be clear that only event $[\bar{R}, \tilde{R}]$ is common knowledge among speculators—no matter how small $\varepsilon$ might be. An event $E \subset [\bar{R}, \tilde{R}]$ is nth-order mutual knowledge at $R \in E$ only if $E \supseteq [\bar{R} - 2n\varepsilon, \bar{R} + 2n\varepsilon] \cap [\bar{R}, \tilde{R}]$, which means that there is always some $n$ for which the last inclusion fails to hold. Hence, small departures from common knowledge may lead to very different results. Indeed, with imperfectly observed reserves a speculator must predict the behavior of speculators receiving signals who are an $\varepsilon$ away from this speculator, which in turn depends on their beliefs about the behavior of speculators who are an $\varepsilon$ away from them, and so on. Therefore, a small seed of noise may infect the whole range of signals.

A strategy for a speculator is now a function from the set of signals to the set of actions. Let $\pi(x)$ denote the proportion of speculators short selling the currency after getting signal $x$. Adding up across signals, under the stock of reserves $\bar{R}$ we get the aggregate amount of short selling:

$$s(\bar{R}, \pi) = \frac{1}{2\varepsilon} \int_{\bar{R}-\varepsilon}^{\bar{R}+\varepsilon} \pi(x) \, dx.$$

For given $\pi$, the peg is abandoned in the event:

$$A(\pi) = \{ R \mid s(\bar{R}, \pi) + s_c > R \}.$$

For a speculator who receives signal $x$, let $u(x, \pi)$ be the expected payoff from short selling one unit of the currency:

$$u(x, \pi) = \frac{1}{2\varepsilon} \int_{[\bar{R}-\varepsilon,x+\varepsilon]} \frac{R}{s(\bar{R}, \pi) + s_c} \, d\bar{R} - c.$$

(3)

An equilibrium of the game occurs if $\pi(x) = 1$ for $u(x, \pi) > 0$, and $\pi(x) = 0$ for $u(x, \pi) < 0$. An equilibrium is called a threshold equilibrium if $\varepsilon > 1 - s_c - s_c/s_c$.
there is $R^*$ such that: (i) The peg is abandoned for all $R < R^*$, and (ii) the peg survives for all $R \geq R^*$.

4. Results

Functions $\pi$ and $s$ will both have particularly simple forms in a threshold equilibrium. This is very convenient to show existence and uniqueness of equilibrium, and to perform some quantitative comparative statics exercises. Our proof of existence and uniqueness of a threshold equilibrium has to cope with the lack of strategic complementarity. This also makes that our equilibrium strategy may not be the only one surviving iterated deletion of strictly dominated strategies (Carlsson and Van Damme, 1993; Milgrom and Roberts, 1990; Morris, 2008).

4.1. Existence and uniqueness of equilibrium

Suppose that $R^*$ defines a threshold equilibrium. For all $x \leq R^* - \varepsilon$ we must have $u(x, \pi) > 0$ because a speculator receiving signal $x$ believes that the peg will be abandoned with probability one. Likewise, for all $x \geq R^* + \varepsilon$ we must have that $u(x, \pi) = -c$. Moreover, $u(x, \pi)$ is strictly decreasing in $x$ within $(R^* - \varepsilon, R^* + \varepsilon)$. Indeed, as we move to the right, the integral in (3) adds up states in which the payoff is $-c$ and leaves off states in which it is positive. By the continuity of the integral, there is a unique $x^*$ fulfilling $u(x^*, \pi) = 0$.

Therefore, modulo sets of measure zero, in any threshold equilibrium $\pi$ takes the form

$$I_x(y) = \begin{cases} 1 & \text{if } y \leq x \\ 0 & \text{if } y > x. \end{cases}$$

(4)

If $\pi = I_x$, then $s(R, \pi)$ is equal to one if $R \leq x - \varepsilon$, and equal to zero if $R > x + \varepsilon$; moreover, it decreases at rate $1/2\varepsilon$ in between. In short:

$$s(R, I_x) = \begin{cases} 1 & \text{if } R \leq x - \varepsilon \\ \frac{1}{2} - \frac{1}{2\varepsilon} (R - x) & \text{if } x - \varepsilon < R \leq x + \varepsilon \\ 0 & \text{if } R > x + \varepsilon. \end{cases}$$

Consequently, event $A(\pi)$ becomes $A(I_x) = [R, \rho(x))$, with

$$\rho(x) = \frac{1}{1 + 2\varepsilon} [x + (1 + 2s_e)\varepsilon].$$

It follows that every $x$ with $u(x, I_x) = 0$ characterizes a threshold equilibrium. We shall show that there is exactly one such $x$. Considering $u(x, I_x)$ as a function of $x$ alone, we see that if $\varepsilon$ is not too big this function is positive at the lower end of the set of signals and negative at the upper end. As one moves to the right, two opposite effects are in action: (i) More speculators are required to cause a devaluation; and (ii) more reserves accrue to each attacker if a devaluation sets off. The first effect is not present in models of bank runs (Goldstein and Pauzner, 2005), and the second effect is not present in currency speculation models without international reserves (Morris and Shin, 1998).

**Proposition 1.** Assume that function $h : (0, 1) \rightarrow \mathbb{R}_+$ is twice continuously differentiable with bounded derivatives. Let condition (1) be satisfied for all $R$ in $[R, R]$. Then, there is a unique threshold equilibrium with $R^* = h(x^*)$. Every speculator receiving a signal $x < x^*$ will attack the peg, and every speculator receiving a signal $x > x^*$ will not attack the peg.

Strictly speaking, there is a continuum of threshold equilibria which only differ in a set of measure zero (at $x^*$). Note that existence and uniqueness of the threshold equilibrium (Proposition 1) does not require any shape restrictions on $h$ as long as condition (1) is satisfied. In the limiting case of no uncertainty, the threshold equilibrium is easy to compute:

**Corollary 1.** As $\varepsilon$ goes to zero, the threshold equilibrium $x = x^*$ is obtained as the solution to the following equation:

$$\int_{R}^{1} h(z) \frac{dz}{z^2} = \frac{c}{x}.$$

Hence, as $\varepsilon$ goes to zero, for the constant payoff function $h(z) = \delta$ we get

$$x^* = 1 + s_e - \frac{c}{\delta}.$$

(5)

**Example 1 (Multiple and non-threshold equilibria).** Consider the function

$$h(z) = \begin{cases} 1 & \text{if } 0 < z < \zeta \\ \left(\frac{2z - \zeta}{1 - \zeta}\right)^2 & \text{if } \zeta \leq z < 1, \end{cases}$$

(6)

A sufficient condition is $2\varepsilon < \min\{s_e - R, R - (1 + s_e)\}$. 

$$4$$
Proposition 2. Let \( h(z) = \delta z \). Then, the threshold equilibrium in Proposition 1 is the only equilibrium.

4.2. Iterated dominance

A remarkable property of the model of Morris and Shin (1998) is that the equilibrium strategy is the only one surviving the iterated deletion of strictly dominated strategies. This property is a direct consequence of global strategic complementarity, but it does not hold for more general payoff structures. Shorting the currency is a dominant strategy for a speculator receiving a signal below \( s_e - \varepsilon \) because the belief is that the peg will certainly be abandoned. Of course, speculators receiving signals above \( s_e - \varepsilon \) understand that \( \pi(x) = 1 \) for all \( x < s_e - \varepsilon \). Hence, some speculators to the right of \( s_e - \varepsilon \) may want to attack the peg as well. More generally, we are interested in the lowest \( x \) in which a speculator can expect zero payoff from shorting the currency—under the presumption that \( \pi(y) = 1 \) for all \( y < x \).

Proposition 3. Under the conditions of Proposition 1, not attacking the peg does not survive the iterated deletion of strictly dominated strategies for all signals \( x < x^* \).

Proposition 3 is a strengthening of standard results in the global games literature that build on the assumption of strategic complementarity in players’ actions. In the proof of this proposition we show that if all speculators to the left of speculator \( x \) attack, then this speculator gets the least from attacking when all speculators to his right do not attack. With a weakly decreasing \( h \) the reverse argument is true if \( \pi(y) = 0 \) for all \( y > h \); then one can show that attacking the peg does not survive the iterated deletion of strictly dominated strategies for all signals \( x > x^* \), and hence there is a unique equilibrium. But iterated dominance does not yield a unique outcome if the payoff from currency depreciation decreases with the mass of speculators attacking the currency.

Proposition 4. Let \( h'(z) \geq 0 \) for \( 0 < z < 1 \) and \( \lim_{z \to 1^-} h'(z) > 0 \). Under the conditions of Proposition 1, there is a maximal \( x^* > x^\dagger \) such that attacking the peg survives the iterated deletion of strictly dominated strategies for all signals \( x \leq x^\dagger \).

Therefore, attacking is compatible with common knowledge of rationality among speculators for all signals \( x \leq x^\dagger \). We also have a closed-form expression for \( x^\dagger \) in the limiting case of no uncertainty.

\[
\zeta = \frac{s_e - 2\varepsilon}{1 + s_e}.
\]

Payoff function \( h \) is depicted in Figure 2, as well as the the corresponding utility \( u(x, I_x) \) for the limiting case in which \( \varepsilon \) goes to zero, and parameter values \( s_e = 3 \) and \( c = 0.15625 \). Note that function \( h \) is U-shaped within the relevant region and violates condition (1) for \( R \) in [3.3024, 3.6976]. In the limit (as \( \varepsilon \) goes to zero) there are three threshold equilibria characterized by points \( x_1^\dagger \approx 3.3254, x_2^\dagger \approx 3.5126, \) and \( x_3^\dagger \approx 3.6662 \).

Moreover, \( u(x, 1 - I_x) = u(x, I_x) \) as \( \varepsilon \) goes to zero. Therefore,

\[
\pi(x) =
\begin{cases}
1 & \text{if } x \leq x_1^\dagger \\
0 & \text{if } x_1^\dagger < x \leq x_2^\dagger \\
1 & \text{if } x_2^\dagger < x \leq x_3^\dagger \\
0 & \text{if } x > x_3^\dagger 
\end{cases}
\]

defines a non-threshold equilibrium.

We now consider some cases in which the threshold equilibrium is the only equilibrium. Under the assumption of global strategic complementarity, one can readily substantiate this uniqueness result by well-known methods. Therefore, for a constant payoff function \( h(z) = \delta \) or for a decreasing payoff function \( h(z) \) there exists a unique equilibrium. Uniqueness also holds for our global games version of Obstfeld’s example.

Figure 2: Functions \( h \) and \( u(x, I_x) \) of Example 1. Parameter values: \( s_e = 3 \) and \( c = 0.15625 \).
Corollary 2. As \( \varepsilon \) goes to zero, \( x^\dagger \) converges to the expression
\[
x^\dagger = 1 + s_e - \frac{c}{\lim_{z \to 1^-} h(z)}.
\]

Combining (5) and (7) we see that \( x^\ast = x^\dagger \) under either \( c = 0 \) or a constant \( h(z) = \delta \). If \( c > 0 \) and \( h(z) \) is increasing then there is a unique threshold equilibrium \( x^\ast \) but a non-degenerate interval of signals \([x^\ast, x^\dagger]\) surviving iterated deletion of strictly dominated strategies.

4.3. Comparative statics

Uniqueness of a threshold equilibrium is quite convenient for the numerical computation of the model and to perform comparative static exercises. As discussed in the introduction, for policy purposes it seems critical to develop a quantitative theory of currency crises. We now explore how changes in parameter values may affect the required amount of reserves.

Proposition 5. Under the conditions of Proposition 1, we have
\[
\frac{d(R^\ast - s_e)}{dc} = \frac{\rho(x^\ast)}{(1 + s_e) h \left( \frac{x^\ast - c}{1 + s_e} \right) - c} < 0
\]
and
\[
\frac{d(R^\ast - s_e)}{ds_e} = \frac{1}{1 + 2\varepsilon} \left[ (x^\ast - \varepsilon) h \left( \frac{x^\ast - \varepsilon}{1 + s_e} \right) + 2\varepsilon c \left( \frac{x^\ast - \varepsilon}{1 + s_e} \right) - c \right].
\]

This latter expression is positive (negative) if \( h \) is increasing (decreasing).

We may interpret \( \varepsilon \) as a measure of the lack of transparency in the conduct of the monetary policy, and \( c/\delta \) as a measure of barriers to capital flows over the payoff from a devaluation. Our numerical exercises will be concerned with the response of quantity \( R^\ast - s_e \) to parameter values. Observe that \( R^\ast - s_e \) is the required level of international reserves to fight speculation. This quantity can also be reinterpreted as the proportion of states in which the peg is abandoned, since the aggregate amount of currency available for speculation has been normalized to one.

Figure 3 displays several plots of \( R^\ast - s_e \) as a function of \( \varepsilon \) for \( s_e = 3 \) and different values of the ratio \( c/\delta \). We have chosen \( c/\delta \leq 1/2 \) to fulfill our parameter’s restrictions \([1] \) and \( \delta < s_e - 2\varepsilon \). The solid lines of Figure 3 refer to the case in which \( h(z) = \delta z \); that is, in the event of a devaluation the available stock of international reserves is prorated among the speculators attacking the currency. The dashed lines of Figure 3 refer to the case in which \( h(z) = \delta \); that is, the payoff from a devaluation does not depend on the level of reserves or the mass of speculators attacking the currency. Each of these lines appears in the figure with the associated transaction cost \( c/\delta = 0, 0.1, 0.2, 0.3, 0.4, \) and \( 0.5 \). Thus, the top line corresponds to \( c/\delta = 0 \), for both functions \( h(z) = \delta z \) and \( h(z) = \delta \), the next line below corresponds to \( c/\delta = 0.1 \), and so on. If the payoff function is a convex combination of functions \( h(z) = \delta z \) and \( h(z) = \delta \), then the numerical results seem to fall in the intermediate range determined by these two functions. Figure 4 reports the same experiments for a decreasing payoff function \( h(z) = \delta (2 - z) \); This payoff function exemplifies a situation in which all exchanges are honored and the depreciation rate goes down with our measure of speculative pressure \( z \).

There are three main results to be highlighted. First, for a small transaction cost \( c/\delta \) the degree of transparency of monetary policy does not really matter. More specifically, \( c/\delta \) arbitrarily close to zero would give rise to the prototypical case of a “one-sided bet”: The degree of transparency of monetary policy becomes irrelevant to stop speculators from shorting the currency.
As a matter of fact, in our quantitative exercises the transparency of monetary policy seems to have relatively little weight for all $c/\delta \leq 0.2$. Second, the amount of required reserves to fight speculation $R^s - s_e$ drops linearly with parameter $c/\delta$. For instance, if $c/\delta$ is moved from $c/\delta = 0$ to $c/\delta = 0.30$ then $R^s - s_e$ goes down by 30 percent. We established this result in (5) for $\varepsilon$ approaching 0, and it holds approximately true over the domain of $\varepsilon$ for all our numerical experiments. Of course, this linear dependence may stem from the assumptions of uniform beliefs, the shape of $h$, and the risk neutrality of speculators. Third, for large values of $c/\delta$, an increase in the transparency of monetary policy tends to increase (decrease) the required amount of international reserves $R^s - s_e$. Figure 3 considers increasing payoff function $h(z) = \delta z$, and so it is not desirable to have a fully transparent monetary policy. Figure 4 considers decreasing payoff function $h(z) = \delta(2 - z)$. Then, it is desirable to have a fully transparent monetary policy. In the boundary case of these two families of payoff functions we have constant function $h(z) = \delta$, where the degree of asymmetric information has no influence on the required level of reserves $R^s - s_e$. We remark that these quantitative effects of parameter $\varepsilon$ seem minor—even for sizable values of $c/\delta$.

Our results may seem controversial and highlight the need for further quantitative work in this area. Numerous writers and international institutions like the IMF have advocated for transparency in the monitoring of international currency reserves. For instance, Morris and Shin (1998, p. 595) write on the same issue:

Above all, our analysis suggests an important role for public announcements by the monetary authorities, and more generally, the transparency of the conduct of monetary policy and its dissemination to the public. If it is the case that the onset of currency crises may be precipitated by higher-order beliefs, even though participants believe that the fundamentals are sound, then the policy instruments which will stabilize the market are those which aim to restore transparency to the situation, in an attempt to restore common knowledge of the fundamentals.

Our quantitative exercises suggest a limited role for the transparency of central bank interventions to curb speculative attacks. Moreover, the direction of the effect of an increase in heterogeneity of beliefs is determined by the shape of the payoff function $h$—the degree of strategic complementarity (or substitutability) that this payoff function entails [cf. Figures 3 and 4]. Thus, it is only desirable to have a transparent economic policy under the presumption of global strategic complementarity in players’ actions.

In recent times we have witnessed new political trends towards the introduction of restrictions to global finance as manifested by various reactions of the IMF, the G-20, and the European Union. Hence, the quantitative importance of transaction costs to deter speculation is also a topic of current interest which is naturally addressed in the present framework. Given that speculators are risk neutral in all our quantitative experiments, the pertinent variable is actually the ratio of the transaction cost over the payoff from depreciation, $c/\delta$. Roughly speaking, in our model this ratio has a one-to-one linear effect on the required level of international reserves. This linear effect seems therefore quite limited to stop speculation at the start of a currency crisis. For our sharing rule $h(z) = \delta z$, a high transaction cost becomes more effective when the monetary policy is less transparent because under diversity of beliefs the payoff from speculation goes down.

Finally, these results seem to be robust to a weakening of the solution concept. We know from Proposition 4 that $R^s = \rho(x^s)$ is the sup over the set of states in which the peg may be broken, provided there is common knowledge of rationality among speculators. Quantity $R^s - s_e$ thus represents the
minimum amount of reserves preventing speculation without requiring speculators' beliefs about others' actions to be correct. (This seems an adequate way of modeling a situation in which the players participate infrequently.) Corollary 1 and Corollary 2 characterize the difference between \( R^l - s_e \) and \( R^* - s_e \) for the limiting case \( \varepsilon = 0 \). From further computational work not reported here, this difference seems to be robust to changes in heterogeneity of beliefs. In conclusion, the difference \( R^l - R^* \) is nearly constant as \( \varepsilon \) varies, and is nearly zero for small transaction costs \( c/\delta \).

5. Extensions

There are certain extensions of considerable economic interest that can readily be accommodated within the present framework. Our model contains a relatively general payoff function \( h \), which can accommodate more general utility functions, and pick up risk aversion effects. Also, the transaction cost may depend on \( z \) (e.g., it may be proportional to \( z \)). Again, a variable transaction cost can easily be accommodated under our general specification of \( h \).

If the payoff function is a constant function \( h(z) = \delta \), then the payoff from speculation is not affected by the excess quantity of currency supplied. If the payoff function is of the form \( h(z) = \delta(z)z \), then the gains from speculation are inversely related to the relative excess quantity of currency supplied (variable \( z \)). The deprecation factor \( \delta \) is endogenously determined by variable \( z \). This latter payoff function \( h(z) = \delta(z)z \) is a version of Obstfeld (1996) with endogenous depreciation \( \delta \). If \( \delta(z) \) is decreasingly monotone it means that the size of the devaluation grows with the relative amount of the excess supply of currency. The existence and uniqueness of a threshold equilibrium holds under general assumptions. But ruling out other forms of equilibria may require further restrictions.

Our analysis of the previous section makes clear that the relevant threshold value to attack the currency is conformed by variable \( R - s_e \). Hence, our model could be reinterpreted as one of full certainty in the amount of reserves \( R \) but with uncertainty on the underlying excess supply \( s_e \) corresponding to the prevailing exchange rate \( \varepsilon \). Consequently, even if we consider a fully transparent monetary policy, we cannot eliminate the noise in other fundamentals of the economy which determine the effective amount of reserves to fight speculation, \( R - s_e \).

We now pass on to discuss some other extensions which require significant changes to our original framework of currency speculation.

5.1. Currency bands

Rather than a fixed exchange rate regime, we may allow the currency to fluctuate within certain margins. In this more general environment, there are two state variables to consider: The effective level of reserves \( R - s_e \) to fight speculation and the distance of the peg from the currency floor, say \( \varepsilon \). In fact, these two state variables can be aggregated into a single one under the following simple extension of our model in which the currency floor \( \varepsilon \) must be surpassed before the volume of international reserves gets depleted.

Let \( e_0 > \varepsilon \) be an initial currency value. In the event of a speculative attack in which the government cannot sustain the currency, international reserves are only sold if the ensuing exchange rate is less than or equal to the floor value \( \varepsilon \). That is, the government would trade the full stock of available reserves only if the exchange rate falls below \( \varepsilon \). The move from \( e_0 \) to \( \varepsilon \) puts downward pressure on both the excess currency supply, \( s_e \), and the gains from speculation, \( \delta \). Therefore, barring other dynamic considerations discussed below, a currency band dissuades speculators because both the excess currency supply and the gains from speculation must be calculated from the price floor \( \varepsilon \) rather than from the existing exchange rate \( e_0 \).

5.2. Large traders, endogenous generation of information, and sequential games

Corsetti, Dasgupta, Morris, and Shin (2004) extend the above framework of Morris and Shin (1998) to two types of agents (a large trader and a continuum of small traders). Agents may be allowed to move sequentially. This is an important extension: If the actions of the large trader can be observed then a small agent could mimic those actions. As illustrated in Figures 3 and 4, however, in our simple model revelation of information and strategic behavior may only be relevant if transaction costs are sizable. Indeed, as the transaction costs converge to zero, a currency attack occurs in equilibrium for \( R < 1 + s_e \); e.g., see (5). Thus, as we approach the limiting case of a “one-sided bet” it follows that a speculator would only be pursued by the gains from currency depreciation regardless of the structure of information.

As Morris and Shin (2004) and Angeletos, Hellwig, and Pavan (2006) have
pointed out, uniqueness of equilibrium may not be robust to more general information structures that allow for private and public information. Multiple equilibria may stem from self-fulfilling expectations which generally occur when the public signal is very accurate. Coordination can be justified under endogenous generation of information, as public choices convey information affecting traders’ behavior.

Dynamic global games of currency speculation (Angeletos, Hellwig, and Pavan, 2007) are also of great relevance for applied work. In these games the characterization and computation of equilibria may become quite complex, which poses major challenges to economic theorists. In a dynamic setting, transaction costs may be quite large, since those costs could be associated with the interest rate differential of holding a weak currency for a few periods. A speculator is then faced with the decision as to when to attack the currency. There is therefore an incentive to preempt other speculators before the currency drops value. The similarities of this game with that of Abreu and Brunnermeier (2003) are quite obvious, and so the assumption of global strategic complementarity seems quite restrictive. Therefore, our present results may be of further interest for dynamic theories of speculation.

6. Concluding Remarks

In this paper we study a global game of currency speculation composed of a government and a continuum of agents. The government holds a stock of international reserves to sustain a currency peg, and the agents can short one unit of the domestic currency to bring about a currency attack. The payoff from currency depreciation is assumed to depend on the amount of speculators shorting the currency. This sharing rule presupposes that the government does not have an unlimited stock of reserves to defend the peg. Incidentally, this rule may also pick up other effects from endogenous currency depreciation. Speculators hold certain beliefs about the amount of international reserves held by the government. Then, as we increase the perceived amount of international reserves, two opposite effects may occur in our global game: (i) More speculators are required to cause a devaluation; and (ii) more reserves would accrue to each attacker if a devaluation sets off. The first effect is not present in models of bank runs (Goldstein and Pauzner, 2005), and the second effect is not present in currency-speculation models without international reserves (Morris and Shin, 1998).

As we do not impose any shape restrictions on the payoff from devaluation, the assumption of strategic complementarity in players’ actions fails to be satisfied. We show existence and uniqueness of a threshold equilibrium under very general conditions. Ruling out existence of other equilibria may require more demanding assumptions. We have established this latter uniqueness result for an important class of affine payoff functions. The threshold equilibrium is then computed by standard approximation methods for various parameter values.

Two main results emerge from our study of these general payoff functions. First, the threshold equilibrium value \( x^* \) is not the only solution of the game that survives iterated deletion of strictly dominated strategies. Indeed, even as the degree of uncertainty converges to zero there is a unique threshold equilibrium and a non-degenerate interval of serially undominated strategy profiles. This result requires positive transaction costs and an increasing payoff function \( h \), but does not fundamentally depend on the uniform distribution of beliefs. Hence, our results critically depart from the well-established equivalence in supermodular games (Milgrom and Roberts, 1990; Morris, 2008) between Nash equilibrium and serially undominated strategy profiles. While strategic complementarity (or zero transaction costs) seems indispensable to insure this equivalence result, existence and uniqueness of the threshold equilibrium does not require any shape restrictions on the payoff functions. As a matter of fact the set of serially undominated strategy profiles can take the form \([x^*, x^+]\) for \( x^+ > x^* \). That is, not attacking the peg does not survive the iterated elimination of strictly dominated strategies for all signals \( x \) below the threshold equilibrium value \( x^* \). This property reflects a strategic complementarity in players’ actions. Indeed, as more speculators attack the currency, an increase in the set of states in which the peg breaks always dominates the substitutability effect embedded in payoff function \( h \). But if the transaction cost is positive and the payoff from speculation decreases with the excess supply of currency, then there is a non-degenerate interval of signals \( x \in [x^*, x^+] \) in which attacking the peg survives iterated elimination of strictly dominated strategies.

Second, numerous writers and international institutions like the IMF have advocated for minimizing the deleterious effects of asymmetric information originating from heterogeneity of beliefs in the volume of reserves or noisy central bank interventions. Our numerical exercises show that the optimal degree of transparency of the economic policy depends on the shape of the
payoff function. Thus, if there is global strategic complementarity in players' actions, then it is optimal to have a fully transparent economic policy (Morris and Shin, 1998). But if strategic substitutability prevails in the region of a successful attack then a noisy economic policy may be more effective to dissuade speculation. Indeed, under strategic substitutability, with an added amount of noise the pivotal speculator \( x^* \) faces a broader range of states \( x \) with lower payoff from attacking the peg, whereas the payoff from not attacking the peg remains constant.

In our model, however, changes in the degree of asymmetric information seem to have mild quantitative effects on the required level of reserves \( R - s_e \). In fact, the heterogeneity of beliefs would only count when transaction costs are large. As the transaction cost converges to zero we get the the limiting case of a “one-sided bet”. That is, attacking the currency becomes a weakly dominant strategy regardless of the structure of asymmetric information.

In summary, our quantitative results point at the inherent instability of a fixed exchange regime: If the borrowing capacity of speculators is greater than the available amount of international reserves then in equilibrium the government is not able to sustain the peg because speculators will attack the currency. This premise is approximately true for low transaction costs and appears to be robust to changes in the degree of asymmetric information. High transaction costs provide some mitigating effects, but for policy purposes these costs may need to be implausibly high. There is therefore an important role to be played by international policy cooperation to economize on the optimal quantity of international reserves.

From an empirical point of view, our model suggests that large currency devaluations should originate from shocks in the level of international reserves, equilibrium exchange rates, transaction costs, and asymmetry of information. As a matter of fact, these results are in line with the empirical findings of Sachs, Tornell, and Velasco (1996) who use a rather simplified model of financial crisis. According to these authors, misbehavior of some of these primitive variables is usually a necessary condition for a crisis. Some other variables such as current account and fiscal deficits play a secondary role. The explanatory power of these latter variables may be understood through their influence on the primitive variables. Along similar lines, Eichengreen, Rose, and Wyplosz (1996) study the influence of contagion on currency crises. They find that contagion can spread more easily to countries which are closely linked by international trade rather than to countries with similar macroeconomic circumstances. Consequently, fundamentals may get deteriorated through various channels producing instability in currency markets.

A related strand of the literature has been concerned with building leading indicators of currency crisis to produce early warning systems. Again, deviations of real exchange rates and ratios of monetary measures to international reserves seem key components of these signaling aggregates (Kaminsky et al., 1998). As Reinhart and Rogoff (2009, p. 279) argue, the success of these leading indicators to predict crises ahead of time has been modest. Indeed, Goldfajn and Valdés (1998) call attention to some common episodes in which markets are unable to foresee exchange rate crises. Hence, crises are often activated by sudden events. Our static model should then capture important aspects of currency speculation.

In contrast to currency speculation, the current European sovereign debt crisis brings to the fore some dynamic aspects of a financial crisis in which the eurozone may temporarily avoid a financial disruption by rolling over the debt. Some analytical issues discussed in this paper are actually at the center of the debate. First, transaction costs may be high. In order to avoid the negative payoff from default, an investor may have to give up high interest rates for a long period of time. Hence, noisy policy interventions along with risk aversion may deter speculative behavior from shorting the debt. Second, bond yield spreads are affected by ECB purchases and the amount of speculators shorting these investments. Hence, the optimal amount of sovereign default should depend on these yields spreads, and the losses from default may be prorated among all investors. Therefore, the assumption of global strategic complementarity in players' actions would also seem overly restrictive to model the European sovereign debt crisis.

\[5\] As discussed in the previous section, a fully transparent economic policy cannot eliminate the noise in other fundamentals of the economy which determine the effective amount of reserves to fight speculation, \( R - s_e \). Hence, the degree of transparency of economic policy has to be analyzed together with the underlying uncertainty of the economic environment.
A. Proofs

Proof of Proposition 1

Let us first write (3) in terms of $z$. Let

$$z(R, \pi) = \frac{R}{s(R, \pi) + s_c}.$$

Consider the set

$$B(z|x, \pi) = \{ R \mid z(R, \pi) \leq z, x - \varepsilon \leq R \leq x + \varepsilon \}.$$  \hspace{1cm} (8)

If $s(R, \pi)$ is decreasing in $R$, then $B$ comprises a (possibly empty) interval of values. More precisely, for $\pi = I_x$ we have

$$B(z|x, I_x) = [x - \varepsilon, \frac{z(1 + 2\varepsilon)\rho(x)}{z + 2\varepsilon}]$$

for all $\frac{x - \varepsilon}{1 - s_c} < z < 1$. The corresponding density of $z$ is the derivative of $B$ with respect to $z$ evaluated at the upper endpoint, normalized by $\frac{1}{z}$, i.e.,

$$f_z(z|x, I_x) = \frac{(1 + 2\varepsilon)\rho(x)}{(z + 2\varepsilon)^2}.$$

Hence, let us rewrite $u(x, I_x)$ as

$$u(x, I_x) = \int_{\varepsilon - s_c}^{1} h(z)f_z(z|x, I_x) \ dz - c.$$  \hspace{1cm} (9)

Differentiating with respect to $x$, we then get

$$\frac{du(x, I_x)}{dx} = \frac{1}{(1 + 2\varepsilon)\rho(x)} \left[ u(x, I_x) + c - (1 + s_c)h \left( \frac{x - \varepsilon}{1 + s_c} \right) \right].$$  \hspace{1cm} (10)

For given $s_c > 0$, conditions (1) for all $R \in [R, \bar{R}]$ and $s_c > 0$ imply that (10) is strictly negative if $u(x, I_x) \leq 0$. Therefore, there is a unique $x$ with $u(x, I_x) = 0$.

Proof of Corollary 1

Take the limit of (9) as $\varepsilon$ goes to zero, and use $u(x^*, I_{x^*}) = 0$.

Proof of Proposition 2

The proof is by contradiction. Assume that there exists some other equilibrium. Let $\pi$ characterize such an equilibrium. Define $\bar{\pi}$ as

$$\bar{\pi} = \sup\{ x \mid \pi(x) > 0 \}.$$

Step 1 [The peg is abandoned for all $R < \rho(\bar{\pi})$]: We know that $s(R, \pi)$ must be weakly decreasing in $[\bar{\pi} - \varepsilon, \bar{\pi} + \varepsilon]$. Since $u(\bar{\pi}, \pi) = 0$, there must be some $R_0$ in $[\bar{\pi} - \varepsilon, \bar{\pi} + \varepsilon]$ at which $s(R_0, \pi) + s_c = R_0$. The expected payoff $u(x, \pi)$ is strictly positive within the interval $[R_0 - \varepsilon, \bar{\pi}]$. Indeed, as we move to the left from its right end, we are excluding states in which the peg survives (and adding some in which it is abandoned). Therefore, $\pi(x) = 1$ for all $x$ in $[R_0 - \varepsilon, \bar{\pi}]$ which, in turn, implies that $R_0 = \rho(\bar{\pi})$. Furthermore, for the same reason, $s(R, \pi)$ must start decreasing at the fastest rate before $\rho(\bar{\pi})$.

Define $\bar{x}$ as

$$\bar{x} = \begin{cases} \bar{\pi} & \text{if } \pi(x) = 1 \text{ for all } x < \bar{x} \\ \sup\{ x < \bar{x} \mid \pi(x) \leq 0 \} & \text{otherwise.} \end{cases}$$

If the equilibrium is not a threshold equilibrium, we must have $\bar{x} < \bar{\pi}$.

Also, by continuity, we must have $u(x, \pi) = u(\bar{x}, \pi) = 0$. We presently show that this is impossible.

Step 2 [If $\bar{x} < \bar{\pi} - 2\varepsilon$, then $\bar{x} = \bar{\pi}$]: If $\bar{x} < \bar{\pi} - 2\varepsilon$, we have that $u(\bar{x}, I_x) = u(\bar{x}, I_{x^*})$, which is zero only if $\bar{x} = x^*$ by Proposition 1. Then, following similar arguments as in Proposition 3 below we get a unique threshold equilibrium.

Step 3 [If $\bar{x} \geq \bar{\pi} - 2\varepsilon$, then $u(\bar{x}, \pi) > u(\bar{x}, \pi)$]: In order to compute $u(\bar{x}, \pi)$ and $u(\bar{x}, \pi)$ we must integrate over the intervals $[\bar{x} - \varepsilon, \bar{\pi} + \varepsilon]$ and $[\bar{\pi} - \varepsilon, \bar{x} + \varepsilon]$. Both intervals overlap and, therefore, we only need to compare the payoff accumulated at both sides of the common subinterval.

Step 3.1: To accomplish this task, we first find a lower bound to the payoff accumulated on the left-hand side subinterval. From (3) we know that such a lower bound can be obtained as follows: (i) Shrinking set $A(\pi)$, and (ii) substituting the denominator $s(R, \pi) + s_c$ by a larger quantity at each state $R$. Let $s_0 = s(\bar{\pi} - \varepsilon, \pi)$ and let $R_0 = \max\{ R < \bar{\pi} - \varepsilon \mid s(R, \pi) + s_c = R \}$. First, we reduce the set $A(\pi)$ by assuming that the peg survives for all states in $[ar{x} - \varepsilon, R_0]$. Second, we find an upper bound for the denominator $s(R, \pi) + s_c$ within $[R_0, \bar{\pi} - \varepsilon]$. We know that $s(R, \pi)$ is weakly increasing in $[\bar{x} - \varepsilon, \bar{\pi} - \varepsilon]$, which means that $s(R, \pi) \leq s_0$ within this interval. The payoff accumulated
in \([x - \varepsilon, x + \varepsilon]\) is bounded below by
\[
\frac{1}{2\varepsilon} \left[ R_{1} \left( R + \frac{R}{2\pi} (R - R_{0}) \right) + \int_{R_{1}}^{\pi - \varepsilon} \frac{R}{s_{0} + s_{e}} \delta \, dR - c \right],
\] (11)
where \(R_{1} = \min\{R_{0} + 2\varepsilon(s_{0} + s_{e} - R_{0}), \pi - \varepsilon\}\). The denominator inside the first integral represents an upward sloping straight line, with initial point \(R_{0}\) on the 45° line until the upper limit \(s_{0} + s_{e}\) is reached; in the second integral the curve becomes flat.

We now show that (11) is decreasing in \(R_{0}\). A sufficient condition for (11) to be decreasing in \(R_{0}\) is that the first integral is so when \(R_{1} = R_{0} + 2\varepsilon(s_{0} + s_{e} - R_{0})\). The derivative of the first summand in (11) in this case is negative if
\[
\log \left( \frac{s_{0} + s_{e}}{R_{0}} \right) \leq \frac{1}{1 - 2\varepsilon}.
\]
On the other hand, the derivative of \(u(x, I_{x})\) with respect to \(x\):
\[
\frac{\partial u(x, I_{x})}{\partial x} = \delta \left[ \log \left( \frac{1 + s_{e}}{\rho(x)} \right) - \frac{1}{1 + 2\varepsilon} \right],
\] (12)
which becomes negative at \(x = x^{*}\). Since \(s_{0} < 1\) and \(R_{0} > \rho(x^{*})\), we have that (11) decreases with \(R_{0}\). Therefore, substituting \(R_{0}\) by a larger number in (11) gives us a lower bound for the payoff accumulated on the left-hand side subinterval.

**Step 3.2:** Let \(\rho(x)\) be the point at which \(s(R_{1} - I_{x}) + s_{e} = R_{0}\):
\[
\rho(x) = \frac{1}{1 - 2\varepsilon} [x - (1 + 2s_{e})\varepsilon].
\]
Note that \(\rho(x) \geq R_{0}\) because \(s_{0} + s_{e} \geq \rho(x)\). Then, the payoff accumulated on the left-hand side is bounded below by
\[
\frac{1}{2\varepsilon} \left[ \int_{\rho(x)}^{\pi - \varepsilon} \frac{\rho(x)}{\frac{1}{2} + \frac{1}{2\pi} (R - x) + s_{e}} \delta \, dR - c \right].
\] (13)
On the other hand, the payoff accumulated on the right-hand side is bounded above by
\[
\frac{1}{2\varepsilon} \left[ \int_{2 + \varepsilon}^{\pi - \varepsilon} \frac{\rho(x)}{\frac{1}{2} - \frac{1}{2\pi} (R - x) + s_{e}} \delta \, dR - c \right].
\] (14)
We have that (13) is greater than or equal to (14) if
\[
\phi(\pi) \log \left( \frac{s_{0} + s_{e}}{\rho(x)} \right) \geq \rho(\pi) \log \left( \frac{s_{0} + s_{e}}{\rho(x)} \right).
\] (15)
But function
\[
x \log \left( \frac{a}{x} \right)
\]
is decreasing if
\[
x \geq a.
\]
In our case
\[
\phi(\pi) \geq \frac{s_{0} + s_{e}}{\rho(x)}
\]
and so
\[
\phi(\pi) \geq \frac{1 + s_{e}}{\rho(x)}
\]
suffices for (15) to be true. Since \(\phi(x) > \rho(x^{*})\) and (12) is negative when evaluated at \(x = x^{*}\), we have \(u(x, \pi) > u(\pi, \pi)\).

**Proof of Proposition 3**

We shall show that \(u(x, I_{x})\) is the least that specular \(x\) can get from attacking if \(\pi(y) = 1\) for all \(y < x\). If \(\pi(y) = 1\) for all \(y < x\) we know from (2) that \(s(R_{1}, \pi)\) is weakly decreasing in \(R\) over \([x - \varepsilon, x + \varepsilon]\), that is, we know that \(B(z|x, \pi)\) in (8) is a nonempty interval for all \(\frac{x - \varepsilon}{1 + T_{n}} < z < 1\). As in the proof of Proposition 1 above, the corresponding density of \(z\) is the derivative of \(B\) with respect to \(z\) evaluated at the upper endpoint, normalized by \(\frac{1}{2\pi}\). The upper endpoint of \(B\) is the unique \(\hat{R} = \hat{R}\) in \([x - \varepsilon, x + \varepsilon]\) that solves \(z(\hat{R}, \pi) = z\). Applying the Implicit Function Theorem to this equation we get:
\[
f_{x}(z|x, \pi) = \frac{1}{2\varepsilon} \frac{d\hat{R}}{dz} = \frac{1}{2\varepsilon} \frac{s(\hat{R}, \pi) + s_{e}}{\rho(\hat{R}, \pi) + s_{e} - \frac{d\rho(\hat{R}, \pi)}{d\hat{R}}}_{R=\hat{R}}
\] (16)
Since \(s(\cdot, \pi)\) is a Lipschitz function, the derivative \(\frac{d\rho(\hat{R}, \pi)}{d\hat{R}}\) exists almost everywhere. The right-hand side of (16) is increasing both in \(s(\hat{R}, \pi)\) and in \(\frac{d\rho(\hat{R}, \pi)}{d\hat{R}}\) when \(\hat{R}\) is increased. We show now that these two quantities are smallest if \(\pi = I_{x}\). First, \(s(\hat{R}, \pi)\) and \(\hat{R}\) are smallest under \(\pi = I_{x}\) for each \(\frac{x - \varepsilon}{1 + T_{n}} < z < 1\) because
the proof proceeds in three steps:

\( u(x, \pi) \geq u(x, I_x) \) for all \( R \) in \( (x - \varepsilon, x + \varepsilon) \) where it exists.

It follows that \( f_2(z|x, \pi) \geq f_2(z|x, I_x) \) a.e. for all \( \frac{x - \varepsilon}{1 + s_e} < z < 1 \). Therefore, \( u(x, \pi) \geq u(x, I_x) \). By Proposition 1, we finally get \( u(x, \pi) \geq u(x, I_x) > 0 \) for all \( x < x^* \).

**Proof of Proposition 4**

Step 1: We first find an upper bound for the payoff of speculator \( x \), provided that \( \pi(y) = 0 \) for \( y > x \). Since we are looking for the maximum payoff, we know there must be some \( R_0 \) in \( (x - \varepsilon, x + \varepsilon) \) so that \( s(R_0, \pi) + s_e = R_0 \). Since \( h'(\cdot) \geq 0 \), we know that \( u(x, \pi) \) is bounded above by

\[
\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{R_0} h \left( \frac{R}{R_0} \right) dR - c = \frac{R_0}{2\varepsilon} \int_0^1 h(z) dz - c, \tag{17}
\]

which is increasing in \( R_0 \). Therefore, the maximum payoff is attained at the maximum \( R_0 \), namely, \( \rho(x) \). Consequently, any \( \pi_x \) satisfying (i) \( \pi_x(y) = 0 \) for \( y \in (x - 2\varepsilon, \rho(x) - \varepsilon) \), and (ii) \( \pi_x(y) = 1 \) for \( y \in (\rho(x) - \varepsilon, x) \), attains the maximum payoff.

Step 2: Function \( u(x, \pi_x) \) is positive if \( x \) is sufficiently small and negative if \( x \) is sufficiently large. Its derivative with respect to \( x \) is:

\[
\frac{du(x, \pi_x)}{dx} = \frac{1}{1 + 2\varepsilon} \left[ u(x, \pi_x) + c - \frac{1 + s_e}{\rho(x)} h \left( \frac{x - \varepsilon}{\rho(x)} \right) \right].
\]

Since \( \rho(x) \leq 1 + s_e \), we have that \( u(x, \pi_x) \leq 0 \) implies that the derivative is strictly negative, which, in turn, implies that there is exactly one \( x^1 \) with \( u(x^1, \pi_x) = 0 \). Therefore, attacking survives the iterated deletion of strictly dominated strategies for all signals \( x \leq x^1 \), and does not survive iterated deletion for all signals above \( x^1 \).

Step 3: Payoffs \( u(x, \pi_x) \) and \( u(x, I_x) \) integrate over the same interval \( [x - \varepsilon, \rho(x)] \). Since \( s(R, \pi_x) \geq s(R, I_x) \) for \( x - \varepsilon \leq R < \rho(x) \), we must have \( z(R, \pi_x) > z(R, I_x) \). Then, \( h'(z) \geq 0 \) for \( 0 < z < 1 \) implies \( h(z(R, \pi_x)) \geq h(z(R, I_x)) \) for \( x - \varepsilon \leq R \leq \rho(x) \). Further, since \( \lim_{x \to \rho(x)} h'(z) > 0 \), the inequality is strict for \( R \) sufficiently close to \( \rho(x) \). Hence, \( u(x, \pi_x) > u(x, I_x) \).

By Proposition 1, \( u(x, I_x) \geq 0 \) for all \( x \leq x^* \), which entails that \( x^1 > x^* \).

\*See the above proof of Proposition 1 using this change of variable.

**Proof of Corollary 2**

From the proof of Proposition 4, we know that \( x^1 \) solves equation

\[
\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x^1} h(z) dz = \frac{c}{\rho(x)}.
\]

Taking the limit as \( \varepsilon \) goes to zero, the result follows by L'Hôpital's rule.

**Proof of Proposition 5**

Both derivatives are obtained from the the Implicit Function Theorem applied to equation \( u(x^*, I_{x^*}) = 0 \). The sign of \( \frac{d[R_{s-e}]}{ds_e} \) follows from (1). Regarding \( \frac{d[R_{s-e}]}{ds_e} \), note from (9) that \( u(x, I_x) \) is bounded below (above) by

\[
(1 + 2\varepsilon)\rho(x) \int_{\frac{x-s_e}{1+s_e}}^{1} h \left( \frac{x-s_e}{1+s_e} \right) dz - c = \frac{1}{1+2\varepsilon} (1+s_e - x + \varepsilon) h \left( \frac{x-e}{1+s_e} \right) - c
\]

if \( h \) is increasing (decreasing). The result is then immediate: \( u(x^*, I_{x^*}) = 0 \) implies that

\[
\frac{(x^* - e) h \left( \frac{x^* - e}{1+s_e} \right) + 2\varepsilon c}{(1+s_e) h \left( \frac{x^* - e}{1+s_e} \right)} - c
\]

is greater (smaller) than one if \( h \) is increasing (decreasing).

**References**


