Location-price equilibrium on trees for a duopoly with negative externality *

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Abstract

This paper presents a location-price equilibrium problem on a tree. Nash equilibrium conditions are presented for a spatial competition model that incorporates price, transport and externality costs. The presented result describes a sufficient condition under which the Nash equilibria on locations and prices are guaranteed. Moreover, the Nash equilibrium on location is a median of the tree. An example is then given to show that there are medians that are not Nash equilibria.

Key words: duopoly, location, price, externality, equilibrium.

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1 Introduction

In general, models in spatial competition involve decisions on location, price and production, made by firms in a spatial market. Equilibrium problems in spatial competition on a network considering location and price or production, have been studied by Lederer and Thisse (1990), Labbé and Hakimi (1991), and Sarkar, Gupta and Pal (1997), among others. Equilibrium models incorporating location and externality cost have been investigated by Brandeau and Chiu (1994a, 1994b).

The paper by Dorta-González, Santos-Peñate, and Suárez-Vega (2002) proposes a spatial competition model in networks incorporating price and externality cost. In this model firms are already located and therefore the location equilibrium is not studied. A regulating agent assigns the demand, taking into account the price, transport and externality costs, and minimizes the joint consumer cost in order to obtain a Pareto optimal allocation. Assuming the Pareto optimal allocation, each firm selects the price, for fixed locations, in order to maximize its profit.

Two different approaches are usually considered when firms have chosen the location and the price. Most models in this context follow Hotelling's formulation (Hotelling, 1929) and use a refinement of the Nash equilibrium. More precisely, firms are supposed to choose location and price, one at a time, in a two-stage process, with the aim of maximizing their own profit. The division into two stages is motivated by the fact that the choice of location is usually prior to the decision on price. In the first stage, firms simultaneously choose their location. Given any outcome of the first stage, firms then simultaneously choose their price in the second stage. The corresponding two-stage solution concept is called a subgame perfect Nash equilibrium (Selten, 1975). It captures the idea that, when firms select their location, they all anticipate the consequences of their choice on price competition.

This paper studies the location-price equilibrium on trees based on the model by Dorta-González, Santos-Peñate, and Suárez-Vega (2002). Therefore, this model involves decisions on location and price in the presence of externalities. Assuming the Pareto optimal allocation, each firm first selects the location of a facility and then selects the price in order to maximize its profit. The goods are assumed to be essential, which means that the demand is perfectly inelastic. Nash equilibrium conditions are presented for a spatial competition model that incorporates price, transport and externality costs. The presented result describes a sufficient condition under which the Nash equilibria on locations and prices are guaranteed. The aim of the first stage is to determine the Nash equilibrium locations considering that the Nash equilibrium prices for each pair of feasible locations are known. A situation is a Nash equilibrium if no firm unilaterally finds it profitable to change. For instance, a pair of locations, x_A and x_B, is a Nash equilibrium if and only if

$$\begin{aligned}\pi_A(x_A, x_B) &= \max_x \pi_A(x, x_B),\\ \pi_B(x_A, x_B) &= \max_x \pi_B(x_A, x),\end{aligned}$$

where π_i represents the profits for firm $i, i = A, B$. In a similar way, a pair of prices, p_A and p_B, is a Nash equilibrium if and only if

$$\begin{aligned}\pi_A(p_A, p_B) &= \max_p \pi_A(p, p_B),\\ \pi_B(p_A, p_B) &= \max_p \pi_B(p_A, p).\end{aligned}$$

The rest of the paper is organized as follows. The location-price equilibrium is discussed in section 2. Section 3 contains an example illustrating a situation where a median of a tree is not a Nash equilibrium on location. Finally, some concluding remarks are presented in section 4.

2 Equilibrium analysis

Let $T(V, E)$ be a tree with node set V and edge set E. Two firms, A and B, located on the tree at points x_A and x_B respectively, provide a product to consumers at nodes $v_k \in V$, $k = 1, 2, ..., n$. The demand at node v_k is λ_k and the total demand is $\Lambda = \sum_{k=1}^{n} \lambda_k$. The demand is perfectly inelastic and it is totally satisfied, therefore $\Lambda = \Lambda_A + \Lambda_B$ where Λ_i is the market share captured by firm i, $i = A, B$. The marginal cost for firm i is assumed to be independent of the quantity supplied and it is denoted by C_i.

Consider also the following notation:

- $t_{ki} = t(d(v_k, x_i))$ is the transport cost from the demand node k to the firm located at x_i,

- p_i is the mill price determined by firm i,

- $e_i > 0$ is the unit externality cost of firm i and $E_i(\Lambda_i) = e_i \Lambda_i$, that is, the externality cost functions are assumed to be linear.

In this section, assuming the Pareto optimal allocation, the location-price equilibrium is obtained. Under the Pareto optimal allocation the study of the location-price equilibrium would be carried out as a two-stage game in a similar way to other equilibrium problems analyzed in the literature (Labbé and Hakimi, 1991; Sarkar, Gupta and Pal, 1997). Firstly, the locations are chosen and then, given these, the prices are determined. To obtain the equilibrium an inverse process is applied; given the locations, the price equilibrium is inferred and then, using this information, the location equilibrium is obtained.

Let x_A and x_B be a pair of fixed locations for the firms,

$$\Delta_k = t_{kB} + p_B - t_{kA} - p_A, \; k = 1, 2, ..., n,$$

$$\Delta_0 = +\infty, \; \Delta_{n+1} = -\infty.$$

The difference of delivered costs (price plus transport cost) between firm B and firm A, Δ_k, is a measure of efficiency. Nodes can be renamed from the point where firm A is the most efficient serving to that where firm B is the most efficient. Without loss of generality, assume that

$$\Delta_1 > \Delta_2 > ... > \Delta_{n-1} > \Delta_n.$$

If $\Delta_k = \Delta_{k+1}$ then the demand nodes k and $k+1$ are aggregated. We also define

$$\begin{aligned} f_j &= \sum_{k=1}^{j} \lambda_k, \; j = 1, 2, ..., n, \qquad f_0 = 0, \qquad f_n = \Lambda, \\ L_j^N &= t_{jB} - t_{jA} + 2\left(e_A + 2e_B\right)\Lambda - 6(e_A + e_B) f_j, \; j = 1, ..., n, \\ T_{j-1}^N &= t_{jB} - t_{jA} + 2\left(e_A + 2e_B\right)\Lambda - 6(e_A + e_B) f_{j-1}, \; j = 1, ..., n. \end{aligned}$$

These values, L_j^N and T_{j-1}^N, allow us to make a division of the real line. Notice that

$$L_n^N < T_{n-1}^N < L_{n-1}^N < \cdots < T_j^N < L_j^N < T_{j-1}^N < L_{j-1}^N < \cdots < T_1^N < L_1^N < T_0^N.$$

Let $j_0 \in \{1, ..., n\}$ such that

$$L_{j_0}^N \leq C_A - C_B \leq T_{j_0-1}^N.$$

Then, from Dorta-González, Santos-Peñate and Suárez-Vega (2002), corollaries 2 and 3, there exists a local equilibrium in $(\bar{p}_A, \bar{p}_B)$,

$$\begin{aligned} \bar{p}_A &= \frac{1}{3}\left[2C_A + C_B + t_{j_0B} - t_{j_0A} + 2\left(e_A + 2e_B\right)\Lambda\right], \\ \bar{p}_B &= \frac{1}{3}\left[C_A + 2C_B + t_{j_0A} - t_{j_0B} + 2\left(2e_A + e_B\right)\Lambda\right]. \end{aligned}$$

Now, using this price equilibrium, the location equilibrium is studied. In order to do this, the following notation is introduced.

Given $v_j \in V$, let

$$V(v_j) = \{v \in V : E(v_j, v) \in E\},$$
$$V(v_j, u) = \{v \in V : \delta_{vv_j} < \delta_{vu}\}, \forall u \in V(v_j),$$
$$\lambda(V(v_j, u)) = \sum_{v \in V(v_j, u)} \lambda_v,$$
$$\eta_j = \min_{u \in V(v_j)} \{\lambda(V(v_j, u))\},$$
$$\gamma_j = \max_{v \in V} \{t(v_j, v)\}.$$

The set $V(v_j)$ is determined by the adjacent nodes to v_j and, for each of these adjacent nodes u, the set $V(v_j, u)$ contains those nodes on the tree which are closer to v_j than u. The minimum total demand of these sets $V(v_j, u)$ for all adjacent nodes is η_j. Finally, γ_j is the maximum transport cost between the nodes of the tree and v_j.

The following assumption establishes a threshold for the marginal costs according to the minimum total demand η_j and the maximum transport cost γ_j. This assumption will be used later in a proposition as a sufficient condition for the existence of a Nash equilibrium on location.

Assumption 1 *There exists $v_{j_0} \in V$ such that*

$$\gamma_{j_0} + 2(e_A + 2e_B)\Lambda - 6(e_A + e_B)\eta_{j_0} \leq C_A - C_B$$
$$\leq -\gamma_{j_0} + 2(e_A + 2e_B)\Lambda - 6(e_A + e_B)(\Lambda - \eta_{j_0}).$$

Note that the inequality

$$\gamma_{j_0} + 2(e_A + 2e_B)\Lambda - 6(e_A + e_B)\eta_{j_0} \leq -\gamma_{j_0} + 2(e_A + 2e_B)\Lambda - 6(e_A + e_B)(\Lambda - \eta_{j_0})$$

occurs if and only if

$$2\gamma_{j_0} \leq 6(e_A + e_B)(-\Lambda + 2\eta_{j_0}),$$

that is

$$\eta_{j_0} \geq \frac{\Lambda}{2} + \frac{\gamma_{j_0}}{6(e_A + e_B)}.$$

As

$$\frac{\Lambda}{2} + \frac{\gamma_{j_0}}{6(e_A + e_B)} > \frac{\Lambda}{2},$$

from condition

$$\eta_{j_0} > \frac{\Lambda}{2}$$

we conclude that *if a vertex* v_{j_0} *satisfies Assumption 1, then* v_{j_0} *is a median of the tree.* Nevertheless, the inverse implication is not true for two reasons.

Firstly, because the difference in marginal costs can be outside the threshold and, secondly, because condition $\eta_{j_0} > \frac{\Lambda}{2}$, which guarantee that v_{j_0} is a median, does not imply $\eta_{j_0} \geq \frac{\Lambda}{2} + \frac{\gamma_{j_0}}{6(e_A+e_B)}$.

Assumption 1 is more restrictive than the existence of a median in the tree. The relative competitiveness, which is expressed by the difference of firms' marginal costs, is one key determinant. In addition to the existence of a median in the tree, Assumption 1 establishes that the difference of marginal costs is not too large, that is, both firms must have a similar competitiveness. On the other hand, the condition $\eta_{j_0} \geq \frac{\Lambda}{2} + \frac{\gamma_{j_0}}{6(e_A+e_B)}$ is satisfied when the maximum transport cost γ_{j_0} is not too large; i.e., the tree is not too deep.

Therefore, Assumption 1 intuitively means that two firms with a similar competitiveness want to locate in a non-deep tree with at least one median.

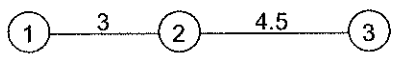

Figure 1: Tree satisfying Assumption 1.

The following example illustrates a situation where Assumption 1 occurs. Figure 1 shows a tree with 3 nodes and demands $\lambda_1 = 2, \lambda_2 = 7, \lambda_3 = 5$. The unit externality costs are $e_A = 0.2$, $e_B = 0.3$. The distances are shown in figure 1 and $t(v_k, v_j) = t(\delta_{v_k v_j}) = \delta_{v_k v_j}$.

Note that v_2 is a median of the tree. Furthermore, if $j_0 = 2$, then $\gamma_{j_0} = 4.5$ and

$$\eta_{j_0} = 9 > \frac{\Lambda}{2} + \frac{\gamma_{j_0}}{6(e_A+e_B)},$$

$$\gamma_{j_0} + 2(e_A + 2e_B)\Lambda - 6(e_A + e_B)\eta_{j_0} = -0.1,$$
$$-\gamma_{j_0} + 2(e_A + 2e_B)\Lambda - 6(e_A + e_B)(\Lambda - \eta_{j_0}) = 2.9.$$

Then, for values of C_A and C_B such that

$$-0.1 \leq C_A - C_B \leq 2.9,$$

Assumption 1 holds, in particular if $C_A = C_B$, for example.

The following proposition establishes that under the previous assumption a Nash equilibrium on location exists and it is a median of the tree.

Proposition *Let $T(V,E)$ be a tree and $t(x,y) = t(\delta_{xy}) = \alpha\delta_{xy}$, $\alpha > 0$. Under Assumption 1, $(\bar{x}_A, \bar{x}_B) = (v_{j_0}, v_{j_0})$ is a Nash equilibrium on locations and v_{j_0}* is a *median of the tree.*

Proof.

It will be proved that each firm maximizes its profit at v_{j_0} when the other firm is located at v_{j_0}. This proof is given as a consequence of a previous result obtained by the same authors in an earlier publication.

(a) Consider $\bar{x}_A = v_{j_0}$ and $x_B \in T$. It will be proved that firm B maximizes its profit when $x_B = v_{j_0}$.

Consider the shortest path between $\bar{x}_A = v_{j_0}$ and $x_B \in T$. The nodes outside this path can be aggregated to the nearest node in the shortest path, resulting in a linear tree. Denoting $u_1 = v_{j_0}$, $u_m = x_B$,

$$\begin{aligned}
L_1^N = L_{j_0}^N &= t_{j_0B} - t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)f_{j_0} \\
&= t_{j_0B} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)f_{j_0} \\
&\leq \gamma_{j_0} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)f_{j_0} \\
&= \gamma_{j_0} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)\eta_{j_0} \\
&\leq C_A - C_B.
\end{aligned}$$

Moreover,

$$\begin{aligned}
T_0^N = T_{j_0-1}^N &= t_{j_0B} - t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)f_{j_0-1} \\
&= t_{j_0B} + 2\,(e_A + 2e_B)\,\Lambda \\
&\geq -\gamma_{j_0} + 2\,(e_A + 2e_B)\,\Lambda \\
&\geq -\gamma_{j_0} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)\,(\Lambda - \eta_{j_0}) \\
&\geq C_A - C_B.
\end{aligned}$$

Since $L_{j_0}^N \leq C_A - C_B \leq T_{j_0-1}^N$, from Dorta-González et all (2002), proposition 8, the profit function for firm B is

$$\begin{aligned}
\pi_B\,(\bar{x}_A, x_B) &= \frac{1}{18(e_A + e_B)}\,(C_A - C_B + t_{j_0A} - t_{j_0B} + 2\,(2e_A + e_B)\,\Lambda)^2 \\
&= \frac{1}{18(e_A + e_B)}\,(C_A - C_B - t_{j_0B} + 2\,(2e_A + e_B)\,\Lambda)^2.
\end{aligned}$$

Note that

$$\begin{aligned}
t_{j_0B} - 2\,(2e_A + e_B)\,\Lambda &\leq \gamma_{j_0} + 2(e_A + 2e_B)\Lambda - 6(e_A + e_B)\eta_{j_0} \\
&\leq C_A - C_B,
\end{aligned}$$

that is

$$C_A - C_B - t_{j_0B} + 2\,(2e_A + e_B)\,\Lambda \geq 0$$

and therefore π_B is maximum when t_{j_0B} is minimum, i.e., when $\bar{x}_B = v_{j_0}$.

(b) Consider $\bar{x}_B = v_{j_0}$ and $x_A \in T$. It will be proved that firm A maximizes its profit when $x_A = v_{j_0}$.

Consider the shortest path between $\bar{x}_B = v_{j_0}$ and $x_A \in T$. The nodes outside of this path can be aggregated to the nearest node in the shortest path, resulting in a linear tree. Denoting $u_1 = x_A$, $u_m = v_{j_0}$,

$$\begin{aligned} L_m^N = L_{j_0}^N &= t_{j_0B} - t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B) f_{j_0} \\ &= -t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)\Lambda \\ &\leq \gamma_{j_0} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)\Lambda \\ &\leq \gamma_{j_0} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)\eta_{j_0} \\ &\leq C_A - C_B. \end{aligned}$$

Moreover,

$$\begin{aligned} T_{m-1}^N = T_{j_0-1}^N &= t_{j_0B} - t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B) f_{j_0-1} \\ &= -t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)\,(\Lambda - \lambda_{j_0}) \\ &\geq -\gamma_{j_0} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)\,(\Lambda - \lambda_{j_0}) \\ &= -\gamma_{j_0} + 2\,(e_A + 2e_B)\,\Lambda - 6(e_A + e_B)\,(\Lambda - \eta_{j_0}) \\ &\geq C_A - C_B. \end{aligned}$$

Since $L_{j_0}^N \leq C_A - C_B \leq T_{j_0-1}^N$, from Dorta-González et all (2002), proposition 8, the profit function for firm A is

$$\begin{aligned} \pi_A\,(x_A, \bar{x}_B) &= \frac{1}{18(e_A + e_B)}\,(C_B - C_A + t_{j_0B} - t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda)^2 \\ &= \frac{1}{18(e_A + e_B)}\,(C_B - C_A - t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda)^2\,. \end{aligned}$$

Note that

$$\begin{aligned} -t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda &\geq -\gamma_{j_0} + 2(e_A + 2e_B)\Lambda - 6(e_A + e_B)\,(\Lambda - \eta_{j_0}) \\ &\geq C_A - C_B, \end{aligned}$$

that is

$$C_B - C_A - t_{j_0A} + 2\,(e_A + 2e_B)\,\Lambda \geq 0$$

and therefore π_A is maximum when t_{j_0A} is minimum, i.e., when $\bar{x}_A = v_{j_0}$.

From (a) and (b) it follows that $(\bar{x}_A, \bar{x}_B) = (v_{j_0}, v_{j_0})$ is a Nash equilibrium on locations. Furthermore, Assumption 1 implies the condition

$$\eta_{j_0} > \frac{\Lambda}{2},$$

and from this we conclude that v_{j_0} is a median of the tree. □

The price equilibrium and the profit functions can be obtained from Dorta-González, Santos-Peñate and Suárez-Vega (2002), proposition 7 and 8, considering that $\bar{x}_A = \bar{x}_B = v_{j_0}$. That is, $(\bar{p}_A, \bar{p}_B)$ with

$$\begin{aligned} \bar{p}_A &= \frac{1}{3}\left[2C_A + C_B + 2\left(e_A + 2e_B\right)\Lambda\right], \\ \bar{p}_B &= \frac{1}{3}\left[C_A + 2C_B + 2\left(2e_A + e_B\right)\Lambda\right], \end{aligned}$$

is a Nash equilibrium in the second stage and the profit functions are

$$\begin{aligned} \pi_A\left(\bar{x}_A, \bar{x}_B, \bar{p}_A, \bar{p}_B\right) &= \frac{1}{18(e_A + e_B)}\left(C_B - C_A + 2\left(e_A + 2e_B\right)\Lambda\right)^2, \\ \pi_B\left(\bar{x}_A, \bar{x}_B, \bar{p}_A, \bar{p}_B\right) &= \frac{1}{18(e_A + e_B)}\left(C_A - C_B + 2\left(2e_A + e_B\right)\Lambda\right)^2. \end{aligned}$$

In the proposition, Nash equilibrium conditions are presented for a spatial competition model that incorporates price, transport and externality costs. The presented result describes a sufficient condition under which the Nash equilibria on locations and prices are guaranteed. An example which shows that there are medians that are not Nash equilibria is given in the next section.

3 Example

This example illustrates a situation where a median of a tree is not a Nash equilibrium on location. Figure 2 shows a tree with 4 nodes and demands $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 3$. The marginal costs are

$$C_A(x) = 10,\ C_B(x) = 5,\ \forall x \in T,$$

and the unit externality costs are $e_A = 0.2$, $e_B = 0.3$. The distances are shown in figure 2 and $t(v_k, v_j) = t(\delta_{v_k v_j}) = \delta_{v_k v_j}$.

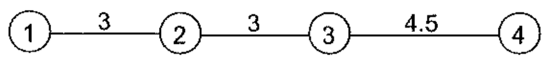

Figure 2: Tree used in the example.

Note that v_2 and v_3 are medians of the tree. Furthermore, if $j_0 = 3$, then $\gamma_{j_0} = 6$ and

$$\eta_{j_0} = 6 = \frac{\Lambda}{2} < \frac{\Lambda}{2} + \frac{\gamma_{j_0}}{6(e_A + e_B)}.$$

Consider $x_A = x_B = v_3$. In this case all the nodes can be aggregated to node v_3. Since $t_{jB} - t_{jA} = 0$, $\forall j$, and

$$-2(2e_A + e_B)\Lambda \leq C_A - C_B \leq 2(e_A + 2e_B)\Lambda,$$

then

$$\pi_A(v_3, v_3) = 22.40, \quad \pi_B(v_3, v_3) = 52.80.$$

Now consider $x_A = v_2$ and $x_B = v_3$. In this case the nodes can be aggregated to the edge $[v_2, v_3]$. Denoting $u_1 = v_3$ and $u_2 = v_2$, results

$$L_1^N = 4.2, \quad T_0^N = 22.2,$$
$$L_1^N \leq C_A - C_B \leq T_0^N.$$

Moreover,

$$\pi_A(v_2, v_3) = 32.87, \quad \pi_B(v_2, v_3) = 39.27.$$

Therefore, (v_3, v_3) is not a Nash equilibrium.

This counterexample proves that medians of trees are not necessarily Nash equilibria.

4 Concluding remarks

In this paper, assuming the Pareto optimal allocation, a location-price equilibrium problem with linear externality costs is investigated. The location equilibrium on trees is studied and under certain conditions, a Nash equilibrium on location exists and it is a median of the tree. As was shown in section 4, there are medians which are not Nash equilibria.

Some extensions to the problem result from considering different hypothesis in the definition of the model, such as, price-elastic demand functions, a leader-follower behavior on price, an oligopolistic market, and other general externality cost functions.

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