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Finding location equilibria for competing firms under delivered pricing

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We address the problem of finding location equilibria of a location-price game where firms first select their locations and then set delivered prices in order to maximize their profits. Assuming that firms set the equilibrium prices in the second stage, the game is reduced to a location game for which a global minimizer of the social cost is a location equilibrium if demand is completely inelastic and marginal production cost is constant. The problem of social cost minimization is studied for both a network and a discrete location space. A node optimality property when the location space is a network is shown and an Integer Linear Programming (ILP) formulation is obtained to minimize the social cost. It is also shown that multiple location equilibria can be found if marginal delivered costs are equal for all competitors. Two ILP formulations are given to select one of such equilibria that take into account the aggregated profit and an equity criterion, respectively. An illustrative example with real data is solved and some conclusions are presented.

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1. Introduction

Major decisions for firms that sell the same type of product and compete for customers are where to locate their facilities and what price to set. The profit each firm gets is affected not only by the location of its facilities and the price that the firm set in the market, but also by the facility locations and the prices set by its competitors. The maximization of profit for each competing firm can be seen as a location–price game, which has been studied since the work by Hotelling (1929). Much existing literature deals with a linear market (see d'Aspremont *et al*, 1979; Osborne and Pitchik, 1987; Gabszewicz and Thisse, 1992), which is in part due to the complexity of solving the associated location problems in other location spaces as the plane or a network (see the survey papers Eiselt *et al*, 1993; Plastria, 2001; Revelle and Eiselt, 2005).

Profit is estimated in most models in this context assuming that customers buy at the cheapest facility. A refinement of the Nash equilibrium by using a two-stage process is taken as solution of the corresponding game. In the first stage, firms simultaneously choose locations. Given any outcome of the first stage, firms then simultaneously choose prices in the second stage. The corresponding two-stage solution is called a subgame perfect Nash equilibrium. It captures the idea that, when firms select a location, they all anticipate the consequences of their choice on price. The division into two stages is motivated by the fact that the choice of location is usually prior to the decision on price. Furthermore, the location decision is relatively permanent whereas the price decision can be easily changed.

The existence of a price equilibrium in the *second stage* of the game depends on the price policy to be considered, among other factors. When each firm sets a factory price equal for all the customers in the market and the buyer takes care of the transportation (f.o.b. or *mill pricing* policy) a price equilibrium rarely exists (see García *et al*, 2004). In this case, the associated location problem has been studied in nonlinear location spaces by taking prices as parameters (see Eiselt, 1992; García and Pelegrín, 2003; Serra and ReVelle, 1999; Zhang, 2001). On the other hand, there frequently exists a price equilibrium when each firm charges a specific price in each market area, which includes the transportation costs *(delivered pricing* policy). The existence of a price equilibrium was shown for the first time by Hoover (1936), who analysed spatial discriminatory pricing for firms with fixed locations and concluded that a firm serving a particular market would be constrained in its local price by the delivery cost of the other firms serving that market. In situations where demand elasticity is 'not too high', the equilibrium price at a given market is equal to the delivery cost of the firm with the second lowest delivery cost. This result was extended later to spatial duopoly (see Lederer and Hurter, 1986; Lederer and Thisse, 1990) and to spatial oligopoly (see García *et al*, 2004;

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Dorta-González *et al*, 2005) for different types of location spaces.

In a duopoly with completely inelastic demand and constant marginal production costs, Lederer and Thisse (1990) show that a location equilibrium exists that is a global minimizer of the *social cost*. The social cost is defined as the total delivered cost if each customer were served with the lowest marginal delivered cost. In oligopoly, the same result is obtained by Dorta-González *et al* (2005), who present a model where firms take location and delivery price decisions along a network of connected but spatially separated markets. Under reasonable assumptions, they show that a location equilibrium can be found at the nodes. Then, a location equilibrium can be determined by global minimization of the social cost, but, to our knowledge, no procedure has been proposed for finding such equilibrium. If demand is price sensitive or marginal production costs are not constant, the socially optimal locations may not be an equilibrium of the location-price game as has been shown in Gupta (1994) and Hamilton *et al* (1989).

This work extends the paper by Dorta-González *et al* (2005), to a multifacility scenario, where each firm sets up multiple facilities, and it presents a solution procedure to find a location equilibrium. For any location space, it is shown that the locations of the firms are in equilibrium if and only if each firm minimizes the social cost with respect to the competitors' fixed locations. If the location space is a network, it is proved that a location equilibrium exists at the nodes. An integer linear programming (ILP) formulation is proposed to find a location equilibrium in discrete location space. In the case that all competitors have the same location candidates and equal marginal delivered costs, it is shown that multiple location equilibria exist. A discussion on the selection of a location equilibrium and an illustrative example are also presented.

The rest of the paper is organized as follows. In Section 2, the notation and the model are introduced for an arbitrary location space and two competing firms. The equilibrium analysis is discussed in Section 3, where a node optimality property is shown when the location space is a network. The ILP formulation for finding a location equilibrium in discrete space is presented in Section 4. Section 5 is devoted to the existence of multiple location equilibria. Two procedures are presented to select one of the multiple equilibria. An illustrative example with real data is also shown. Finally, an extension to oligopoly and some conclusions are presented in Section 6.

2. The location–price game

We consider several firms playing a location-price game in an *arbitrary location space* (eg a line, a network, the plane, etc.) in which there is a set of spatially separated market areas. We assume that markets are aggregated at n demand points (see Francis *et al*, 2002 for demand aggregation). At each demand point a given homogeneous price-inelastic product will be sold by the competing firms. The firms manufacture and deliver the product to the customers, which buy from the firm that offers the lowest price. First the firms have to choose their facility locations in some predetermined location space, and then, once their facility locations are set, the firms will set delivered prices at each demand point. In this way, each firm has to make decisions on location and price in order to maximize its profit. For simplicity, we consider two firms locating several facilities each, but the results obtained can be extended to any number of competing firms, as will be shown in Section 6.

The following notation will be used:

k, $K = \{1, \ldots, n\}$	index and set of markets
q_k	demand in market k
$u = 1, 2$	index of the firm
L^u	set of possible facility locations for firm u
X^u	set of facility locations chosen by firm u, $X^u \subset L^u$
c_x^u	marginal production cost of firm u at location x, $x \in L^u$
t_{xk}^u	marginal transportation cost of firm u from location x to market k
$p_{xk}^u = c_x^u + t_{xk}^u$	marginal delivered cost (or minimum delivered price) of firm u from location x to market k
$p_k^u(X^u)$	$= \min\{p_{xk}^u : x \in X^u\}$, minimum price that firm u can offer in market k

Note that L^u may be a finite set of points (Discrete Location), the points in a network, nodes or points on the edges (Network Location), or a region in the plane (Planar Location).

2.1. Price equilibrium

In this subsection, we show the existence of a price equilibrium for the second stage of the game. It is assumed that customers do not have any preference concerning the supplier and they buy from the firm that offers the lowest price. We consider that, each firm cannot offer a price below its marginal delivered cost and each facility can supply all demand placed on it. Thus, for each market k, once the sets of locations X^u, $u=1, 2$, are fixed, firm u, $u=1, 2$, will set a price greater than, or equal to, $p_k^u(X^u)$. If the two firms offer the same price in market k, the one with the minimum marginal delivered cost can lower its price and obtains all the demand in market k. Then we consider that ties in price are broken in favour of the firm with lower marginal delivered cost. If the tied firms have the same marginal delivered cost in market k, no tie breaking rule is needed to share demand because they will obtain zero profit from market k as a result of price competition.

In the long-term competition, customers in market k will not buy from firm u if $p_k^u(X^u) > \min\{p_k^u(X^u) : u = 1, 2\}$, in such a case its competitors can offer the lowest price. In order to maximize its profit, the firm with the minimum marginal delivered cost in market *k* will set a price equal to the *minimum marginal delivered cost* of its competitor, which in turn is, as result of the price competition process, the price set by its competitor in market *k*. With such prices, none of the two firms will react by changing the price and therefore a price equilibrium is obtained.

Therefore, for any fixed sets of locations, X^1 and X^2, price competition leads to the following equilibrium prices:

$$\bar{p}_k^1(X^1, X^2) = \begin{cases} p_k^2(X^2) & \text{if } p_k^1(X^1) < p_k^2(X^2) \\ p_k^1(X^1) & \text{otherwise} \end{cases}$$

$$\bar{p}_k^2(X^1, X^2) = \begin{cases} p_k^1(X^1) & \text{if } p_k^2(X^2) < p_k^1(X^1) \\ p_k^2(X^2) & \text{otherwise} \end{cases}$$

2.2. Reduction to a location game

We assume that, once the sets of facility locations X^u, $u=1, 2$, are fixed, the firms will set the equilibrium price in each market *k*. The markets in which each firm gets a positive profit are determined as follows:

$$M^1(X^1, X^2)=\{k \in K : p_k^1(X^1) < p_k^2(X^2)\} \quad \text{served by firm 1}$$

$$M^2(X^1, X^2)=\{k \in K : p_k^2(X^2) < p_k^1(X^1)\} \quad \text{served by firm 2}$$

Other markets are served by both firms, but the profit coming from these markets is zero. The profit functions are:

$$\Pi^1(X^1, X^2) = \sum_{k \in M^1(X^1,X^2)} (p_k^2(X^2) - p_k^1(X^1))\, q_k$$

$$\Pi^2(X^1, X^2) = \sum_{k \in M^2(X^1,X^2)} (p_k^1(X^1) - p_k^2(X^2))\, q_k$$

Then the location-price game is reduced to a location game where decisions are on location and $\Pi^u(X^1, X^2)$ is the payoff for player $u, u = 1,2$. This game is studied in the following section.

3. Existence of location equilibria

The *social cost* is the total cost incurred to supply demand to customers if each customer would pay for the product the minimum delivered cost. For any fixed sets of locations, X^u, $u = 1, 2$, the social cost is given by:

$$S(X^1, X^2) = \sum_{k=1}^{n} \min\{p_k^1(X^1), p_k^2(X^2)\}\, q_k$$

Property 1 *If the firms set the equilibrium prices in each market, then*:

$$\Pi^1(X^1, X^2) = \sum_{k=1}^{n} p_k^2(X^2) q_k - S(X^1, X^2)$$

$$\Pi^2(X^1, X^2) = \sum_{k=1}^{n} p_k^1(X^1) q_k - S(X^1, X^2)$$

Proof The payoff for firm 1 can be expressed as follows:

$$\begin{aligned}\Pi^1(X^1, X^2) &= \sum_{k \in M^1(X^1,X^2)} (p_k^2(X^2) - p_k^1(X^1))\, q_k \\ &\quad + \sum_{k \notin M^1(X^1,X^2)} p_k^2(X^2) q_k - \sum_{k \notin M^1(X^1,X^2)} p_k^2(X^2) q_k \\ &= \sum_{k=1}^{n} p_k^2(X^2) q_k - \sum_{k=1}^{n} \min\{p_k^1(X^1), p_k^2(X^2)\} q_k \\ &= \sum_{k=1}^{n} p_k^2(X^2) q_k - S(X^1, X^2)\end{aligned}$$

Thus, the profit obtained by firm 1 is the total cost that would be experienced by its competitor serving the entire market with minimum delivered cost minus the social cost. A similar expression is obtained for firm 2. □

Property 2 $(\overline{X}^1, \overline{X}^2)$ *is a location equilibrium if*:

$$S(\overline{X}^1, \overline{X}^2) \leqslant S(X^1, \overline{X}^2), \qquad \forall X^1$$

$$S(\overline{X}^1, \overline{X}^2) \leqslant S(\overline{X}^1, X^2), \qquad \forall X^2$$

Proof The pair $(\overline{X}^1, \overline{X}^2)$ is a location equilibrium if and only if:

$$\Pi^1(\overline{X}^1, \overline{X}^2) \geqslant \Pi^1(X^1, \overline{X}^2), \qquad \forall X^1$$

$$\Pi^2(\overline{X}^1, \overline{X}^2) \geqslant \Pi^2(\overline{X}^1, X^2), \qquad \forall X^2$$

and from Property 1, these inequalities are equivalent to the following ones:

$$S(\overline{X}^1, \overline{X}^2) \leqslant S(X^1, \overline{X}^2), \qquad \forall X^1$$

$$S(\overline{X}^1, \overline{X}^2) \leqslant S(\overline{X}^1, X^2), \qquad \forall X^2 \qquad \square$$

Property 3 *Any global minimizer of* $S(X^1, X^2)$ *is a location equilibrium.*

Proof It follows from Property 2. □

From Property 3 it follows that we can determine location equilibria by solving the following problem:

$$(P) : \min \{S(X^1, X^2) : X^1 \subset L^1, X^2 \subset L^2\}$$

If L^u, $u=1, 2$, is a region in the plane, problem (*P*) is very difficult to solve and require very complex global optimization techniques. This case is outside the scope of this paper and it will not be considered in the following.

3.1. Location on a network

If L^u, $u = 1, 2$, is the set of points in a network (nodes and points on the edges), in Dorta-González *et al* (2005) it is shown that, if each firm locates a single facility, there exists a set of nodes that is a global minimizer of the social cost.

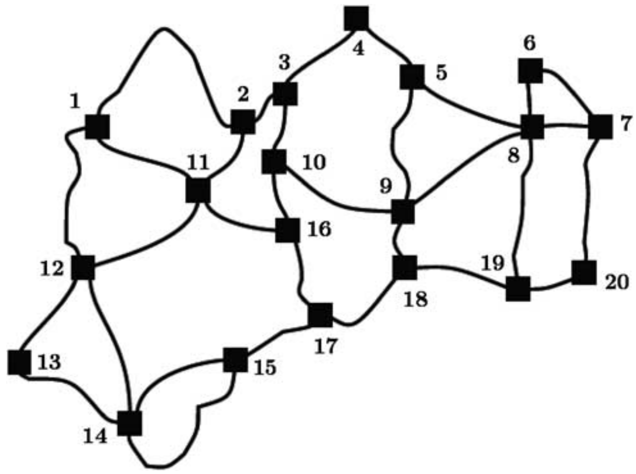

Figure 1 Example of a network.

We extend this result to the case where each firm operates several facilities. Let $N = (V, E, l)$ denote a network, where V is a finite set of nodes, E is the set of edges (pairs of nodes), and l is the function that assigns to each edge a positive number (length, time, cost, ..) associated to the edge. A diagram of a network is shown in Figure 1.

Assumption 1 *For $u = 1, 2$, the marginal production cost, c_x^u, is a positive concave function when x varies along any edge in the network, and it is independent of the quantity produced.*

Assumption 2 *For $u = 1, 2$, the marginal transportation cost, t_{xk}^u, is a positive, concave and increasing function with respect to the distance from x to each market k.*

Property 4 *Under Assumptions 1 and 2, there exists a set of nodes which is a global minimizer of the social cost.*

Proof Let X^1 and X^2 be arbitrary sets of facility locations in the network. If $x \in X^1$ is not a node, then x is in the interior of some edge $e = (a, b) \in E$. Assume that all points in X^1 and X^2 are fixed, but the point x varies in edge e. Under Assumptions 1 and 2, it results that the minimum price to serve market k, $\min \{p_k^1(X^1), p_k^2(X^2)\}$, is a concave function when x varies in edge e. As the sum of weighted concave functions, with non-negative weights, is also concave, it follows that the social cost, $S(X^1, X^2)$, is concave when x varies in edge e, and the other locations are fixed. Therefore, the social cost reaches its minimum value on edge e for $x = a$ or $x = b$. A similar result is obtained if $x \in X^2$ is not a node.

Therefore, if we replace each non-node point in X^1 and X^2 by the corresponding minimizer node of the social cost, we will obtain two sets of nodes V^1 and V^2 for which $S(V^1, V^2) \leqslant S(X^1, X^2)$. Consequently, there exists a set of nodes that minimizes the social cost. □

Thus, the problem of finding a location equilibrium in a network is reduced to the same problem as in discrete space. In the following section we deal with the problem of finding location equilibria in discrete space.

4. Determination of location equilibria in discrete space

In this section, we present an ILP formulation to find location equilibria when L^u, $u = 1, 2$, is a finite set of points. In a network, by Property 4, we can use $L^u = V$, $u = 1, 2$, in order to find location equilibria.

As L^u is finite for $u = 1, 2$, X^u can be represented by a vector x^u with components $x_i^u, i \in L^u$, with 0–1 values, where $x_i^u = 1$ indicates that facility location i is chosen by the firm u, that is $i \in X^u$, and $x_i^u = 0$ means that $i \notin X^u$. We assume that the number of facilities to be located by firms 1 and 2 is r and s, respectively. The number of facilities of the firms are not decision variables, but they are determined in each situation depending on some exogenous factors (budget, regulation, arbitration, etc).

For any fixed location sets x^1 and x^2, we consider the 0–1 variables z_{ik}^u, $i \in L^u$, where $z_{ik}^u = 1$ indicates that firm u serves market k from a facility at location i. If $z_{ik}^u = 0$, it means that market k is not served from a facility of firm u at location i. Then, the social cost is given by solving the following problem in variables z_{ik}^u

$$S(x^1, x^2) = \min \quad \sum_{k=1}^{n} \left(\sum_{i \in L^1} p_{ik}^1 z_{ik}^1 + \sum_{i \in L^2} p_{ik}^2 z_{ik}^2 \right) q_k$$

$$\text{s.t.} \quad \sum_{i \in L^1} z_{ik}^1 + \sum_{i \in L^2} z_{ik}^2 = 1, \qquad \forall k \in K$$

$$z_{ik}^u \leqslant x_i^u, u = 1, 2 \quad \forall k \in K, \quad \forall i \in L^u$$

$$z_{ik}^u, x_i^u \in \{0, 1\}, k \in K, i \in L^u, u \in \{1, 2\}$$

Therefore, the social cost minimization problem becomes:

$$(P) : \min \quad \sum_{k=1}^{n} q_k \sum_{i \in L^1} p_{ik}^1 z_{ik}^1 + \sum_{k=1}^{n} q_k \sum_{i \in L^2} p_{ik}^2 z_{ik}^2$$

$$\text{s.t.} \quad \sum_{i \in L^1} z_{ik}^1 + \sum_{i \in L^2} z_{ik}^2 = 1, k \in K \tag{1}$$

$$z_{ik}^1 \leqslant x_i^1, k \in K, i \in L^1 \tag{2}$$

$$z_{ik}^2 \leqslant x_i^2, k \in K, i \in L^2 \tag{3}$$

$$\sum_{i \in L^1} x_i^1 = r \tag{4}$$

$$\sum_{i \in L^2} x_i^2 = s \tag{5}$$

$$z_{ik}^u, x_i^u \in \{0, 1\}, \; k \in K, \; i \in L^u, u \in \{1, 2\} \tag{6}$$

Once the above problem is solved, the profit of each firm is obtained by taking into account that profit of a firm plus social cost equals total delivered cost of its rival (see Property 1).

Problem (P) can be solved by any standard ILP-optimizer (XPress-Mp, Cplex, ...), however computational difficulties may occur when the number of binary variables is large. To solve problem (P), the constraints $z_{ik}^u \in \{0, 1\}$ in (6) can be relaxed to $z_{ik}^u \geqslant 0$, which makes that large instances of the problem can be solved in small run time. The ILP-optimizers generate one optimal solution $(\bar{x}^1, \bar{x}^2)$ of problem (P), which is a location equilibrium. In some cases, the existence of multiple global minimizers of social cost may be possible, and therefore the existence of multiple location equilibria. In the following, we show the existence of multiple equilibria in a particular case of the location game, and we present two procedures to select one of such equilibria.

5. Existence of multiple location equilibria

When the location space is discrete, social cost minimization is a combinatorial optimization problem that may have multiple global optima and then more than one location equilibrium could exist. In particular, this occurs if firms locate a fixed number of facilities, have a common set of location candidates ($L^1 = L^2$), and the minimum delivered prices that firms can offer are equal ($p_{ik}^1 = p_{ik}^2$, for all i and all k). In such a case, the firms will locate at different sites and the ILP formulation of problem (P) is reduced to the well-known $(r + s)-$*MEDIAN* problem, where r is the number of facilities for firm 1 and s is the number of facilities for firm 2 (see Rolland *et al*, 1996; Avella *et al*, 2007 for some references on this problem). Due to symmetry, it is verified that any partition of the optimal solution set, which contains $r + s$ locations, into two subsets of cardinality r and s, respectively, is a location equilibrium. Therefore, a large number of location equilibria can be obtained ($\frac{(r+s)!}{r!s!}$ combinations).

When more than one location equilibrium are found, the competing firms would agree to select a Pareto optimum equilibrium. Otherwise both firms would obtain less profit than the one obtained by choosing a Pareto optimum. In the following, we discuss the case in which the firms choose an equilibrium that maximizes the aggregated profit. This is justified by the fact that this equilibrium is a Pareto optimum for the firms, and it is not unusual for competitors to reach agreements that benefit both firms. Furthermore, an equity criterion is also considered to select a location equilibrium that guarantees a minimum level of average profit per facility to each competing firm.

5.1. *ILP formulations for selecting a location equilibrium*

In the above mentioned situation, if $\overline{X}$ is an optimal solution of problem (P) (an $(r + s)-$*MEDIAN* problem), any partition of $\overline{X}$ into two subsets, $\overline{X}^1$ and $\overline{X}^2$, of r and s locations in $\overline{X}$, respectively, is a location equilibrium. There exist a plethora of possible criteria to select one of the many partitions of a social cost minimizer (optimal solution of (P)). We will deal with the selection of one partition that maximizes the aggregated profit the competing firms get. The aggregated profit corresponding to a partition $\overline{X}^1$, $\overline{X}^2$ is:

Table 1 Facility locations

Label	*Node*	*Name of city*	*Population*
1	14	Albatera (Alicante)	8633
2	118	Barcelona	1 503 884
3	194	Sta. Coloma de Cervello (Barcelona)	5557
4	208	Letona (Alava)	6072
5	374	Oroso (La Coruña)	5530
6	446	Huetor Tajar (Granada)	8839
7	583	Castro del Rei (Lugo)	5850
8	588	A Fonsagrada (Lugo)	5082
9	604	Alcorcón (Madrid)	153 100
10	630	Madrid	2 938 723
11	632	Manzanares el Real (Madrid)	4547
12	654	Tres Cantos (Madrid)	36 927
13	755	Oviedo (Asturias)	201 154
14	839	Arahal (Sevilla)	18 365
15	875	La Puebla de Cazalla (Sevilla)	10 518
16	881	Sevilla	684 633
17	941	Aielo de Malferit (Valencia)	4155
18	963	Carlet (Valencia)	14 213
19	983	Ontinyent (Valencia)	32 664
20	998	Valencia	738 441
21	1035	Ugao-Miraballes (Vizcaya)	4104
22	1046	Zaragoza	614 905

$$\Pi^a(\overline{X}^1, \overline{X}^2) = \sum_{k=1}^{n} (p_k^1(\overline{X}^1) + p_k^2(\overline{X}^2))q_k - 2S(\overline{X}^1, \overline{X}^2)$$

where $S(\overline{X}^1, \overline{X}^2) = S^*$, S^* being the minimum social cost (optimal value of problem (P)).

Let $p_{ik} = p_{ik}^1 = p_{ik}^2$. If we define the 0–1 variables d_e, $l \in \overline{X}$, where $d_l = 1$ indicates that location l is assigned to firm 1, and $d_l = 0$ means that l is assigned to firm 2; and the continuous variables p_k^u, $u = 1, 2$, that represent the minimum delivered price to serve market k from some facility location of firm u. Then, the problem of finding the partition of $\overline{X}$ that maximizes the aggregated profit can be formulated as:

$$(P^a(\overline{X})): \quad \max \quad \sum_{k=1}^{n} (p_k^1 + p_k^2)q_k - 2S^*$$

$$\text{s.t.} \quad p_k^1 \leqslant p_{lk} d_l + D(1 - d_l), \; k \in K, \; l \in \overline{X} \qquad (7)$$

$$p_k^2 \leqslant p_{lk}(1 - d_l) + D d_l, \; k \in K, \; l \in \overline{X} \qquad (8)$$

$$\sum_{l \in \overline{X}} d_l = r \qquad (9)$$

$$d_l \in \{0, 1\}, \; p_k^1 \geqslant 0, \; p_k^2 \geqslant 0, \; k \in K, \; l \in \overline{X} \qquad (10)$$

where D is a fixed number greater than the maximum marginal delivered price. Constraints (7) guarantee that p_k^1 is not greater than the minimum marginal delivered price of firm 1. Similarly, constraints (8) guarantee that p_k^2 is not greater than the minimum marginal delivered price of firm 2. The number D

Figure 2 Demand points • and facility locations ■.

is used so as constraints (7) and (8) hold for any assignment to variables d_l. In the optimal solution, both variables p_k^1 and p_k^2 are equal to the minimum delivered price in market k of firm 1 and firm 2, respectively. Thus, the optimal value of problem $(P^a(\overline{X}))$ is the maximum aggregated profit the firms can obtain.

If $\overline{d}_l$, $\overline{p}_k^1$, and $\overline{p}_k^2$, is an optimal solution to problem $(P^a(\overline{X}))$, then the optimal partition is:

$$\overline{X}^1 = \{l \in \overline{X} : \overline{d}_l = 1\}$$

$$\overline{X}^2 = \{l \in \overline{X} : \overline{d}_l = 0\}$$

and the maximum aggregated profit is:

$$\Pi^a(\overline{X}^1, \overline{X}^2) = \sum_{k=1}^{n} (\overline{p}_k^1 + \overline{p}_k^2) q_k - 2S^*$$

For the firms to agree on the choice of the location equilibrium previously obtained some equity criterion should be satisfied. For instance, the average profit per facility each firm obtains should be close to the global average profit per facility that is defined as the aggregated profit divided

Table 2 Location equilibria with maximum aggregated profit

r+s	$S(\overline{X})$	r	s	$\overline{X}^1$	$\overline{X}^2$	$\frac{\Pi^1(\overline{X}^1,\overline{X}^2)}{r}$	$\frac{\Pi^2(\overline{X}^1,\overline{X}^2)}{s}$	$\overline{\Pi}^a$
2	6986	1	1	9	3	9367	2980	12 348
3	5414	1	2	3	12,15	2891	5470	13 831
4	4240	1	3	2	11,15,18	1561	4138	13 975
		2	2	11,15	2,18	3136	2101	10 475
5	3244	1	4	8	2,10,15,18	1047	3512	15 097
		2	3	2,18	8,10,15	2003	2423	11 274
6	2549	1	5	7	2,4,10,14,19	977	3104	16 496
		2	4	7,14	2,4,10,19	1259	2428	12 232
		3	3	2,14,19	4,7,10	1714	1644	10 075
7	2223	1	6	7	2,4,6,10,16,17	977	2641	16 821
		2	5	6,16	2,4,7,10,17	935	2048	12 109
		3	4	6,7,16	2,4,10,17	949	2035	10 986
8	1985	1	7	5	2,4,6,10,13,16,17	330	2582	18 405
		2	6	5,13	2,4,6,10,16,17	607	2202	14 428
		3	5	5,13,16	2,4,6,10,17	580	1859	11 035
		4	4	5,6,13,16	2,4,10,17	771	1885	10 626
9	1807	1	8	5	2,6,10,13,16,19,21,22	330	2282	18 583
		2	7	5,13	2,6,10,16,19,21,22	592	1913	14 575
		3	6	5,13,21	2,6,10,16,19,22	596	1591	11 337
		4	5	5,6,13,16	2,10,19,21,22	762	1544	10 770
10	1641	1	9	5	1,2,6,10,13,16,20,21,22	330	2047	18 750
		2	8	5,13	1,2,6,10,16,20,21,22	592	1695	14 741
		3	7	5,13,21	1,2,6,10,16,20,22	596	1388	11 502
		4	6	5,6,13,16	1,2,10,20,21,22	749	1314	10 883
		5	5	5,6,10,13,16	1,2,20,21,22	1039	984	10 116

Table 3 Results for $\lambda = 0.5$

$r+s$	r	s	$\overline{X}^1$	$\overline{X}^2$	$\frac{\Pi^1}{r}$	$\frac{\Pi^2}{s}$	$\lambda\frac{\overline{\Pi}^a}{r+s}$	$\overline{\Pi}^a_\lambda$	$\overline{\Pi}^a$
2	1	1					3087		12 348
3	1	2	3	12,15	2891	5470	2305	13 831	13 831
4	1	3	11	2,15,18	3124	1969	1747	9031	13 975
	2	2	2,18	11,15	2101	3136	1309	10 475	10 475
5	1	4	2	8,10,15,18	1561	3352	1510	14 971	15 097
	2	3	2,18	8,10,15	2003	2423	1127	11 274	11 274
6	1	5	2	4,7,10,14,19	1732	2821	1375	15 837	16 496
	2	4	7,14	2,4,10,19	1259	2428	1019	12 232	12 232
	3	3	2,14,19	4,7,10	1714	1644	840	10 075	10 075
7	1	6	2	4,6,7,10,16,17	1708	2405	1201	16 139	16 821
	2	5	6,16	2,4,7,10,17	935	2048	865	12 109	12 109
	3	4	6,7,16	2,4,10,17	949	2035	785	10 986	10 986
8	1	7	2	4,5,6,10,13,16,17	1708	2096	1150	16 377	18 405
	2	6	6,16	2,4,5,10,13,17	935	1746	902	12 347	14 428
	3	5	5,6,16	2,4,10,13,17	733	1764	670	11 021	11 035
	4	4	5,6,13,16	2,4,10,17	771	1885	664	10 626	10 626
9	1	8	2	5,6,10,13,16,19,21,22	1327	1856	1032	16 174	18 583
	2	7	6,16	2,5,10,13,19,21,22	933	1522	810	12 522	14 575
	3	6	5,6,16	2,10,13,19,21,22	732	1500	630	11 196	11 337
	4	5	5,6,13,16	2,10,19,21,22	762	1544	598	10 770	10 770
10	1	9	2	1,5,6,10,13,16,20,21,22	1327	1668	937	16 340	18 750
	2	8	6,16	1,2,5,10,13,20,21,22	907	1353	737	12 636	14 741
	3	7	5,13,21	1,2,6,10,16,20,22	596	1388	575	11 502	11 502
	4	6	5,6,13,16	1,2,10,20,21,22	749	1314	544	10 883	10 883
	5	5	5,6,10,13,16	1,2,20,21,22	1039	984	506	10 116	10 116

by the total number of facilities $(\Pi^a(\overline{X}^1, \overline{X}^2)/(r+s))$. However, it is possible that, for a partition that maximizes the aggregated profit, one of the firms gets much less profit per facility than its competitor, and therefore some kind of compensation would be required in order to obtain an agreement.

An alternative way of selecting a partition is by including *equity constraints* in the above formulation. The aim of such constraints is to determine a location equilibrium, so that both firms get similar profits per facility, if such equilibrium exists. Let $\overline{\Pi}^a$ denote the maximum aggregated profit, which can be obtained by solving problem $(P^a(\overline{X}))$. For any λ, $0 \leqslant \lambda \leqslant 1$, we consider the following equity constraint: *the average profit per facility each firm obtains is greater than, or equal to*, $\lambda\overline{\Pi}^a/(r+s)$. Then, we can obtain a location equilibrium verifying that constraint, for which aggregated profit is maximum, by solving the following ILP problem:

$$(P^a_\lambda(\overline{X})): \quad \max \quad \sum_{k=1}^{n}(p_k^1 + p_k^2)q_k - 2S^*$$

$$\text{s.t.} \quad p_k^1 \leqslant p_{lk}d_l + D(1-d_l), k \in K, l \in \overline{X} \quad (11)$$

$$p_k^2 \leqslant p_{lk}(1-d_l) + Dd_l, k \in K, l \in \overline{X} \quad (12)$$

$$\sum_{l \in \overline{X}} d_l = r \quad (13)$$

$$\frac{1}{r}\left(\sum_{k=1}^{n} p_k^2 q_k - S^*\right) \geqslant \lambda \frac{\overline{\Pi}^a}{r+s} \quad (14)$$

Table 4 Results for $\lambda = 0.6$

$r + s$	r	s	$\overline{X}^1$	$\overline{X}^2$	$\frac{\Pi^1}{r}$	$\frac{\Pi^2}{s}$	$\lambda\frac{\overline{\Pi}^a}{r+s}$	$\overline{\Pi}^a_\lambda$	$\overline{\Pi}^a$
2	1	1					3704		12 348
3	1	2	3	12,15	2891	5470	2766	13 831	13 831
4	1	3					2096		13 975
	2	2	2,18	11,15	2101	3136	1571	10 475	10 475
5	1	4					1812		15 097
	2	3	2,18	8,10,15	2003	2423	1353	11 274	11 274
6	1	5	2	4,7,10,14,19	1732	2821	1650	15 837	16 496
	2	4	7,14	2,4,10,19	1259	2428	1223	12 232	12 232
	3	3	2,14,19	4,7,10	1714	1644	1007	10 075	10 075
7	1	6	2	4,6,7,10,16,17	1708	2405	1442	16 139	16 821
	2	5	2,17	4,6,7,10,16	1745	1667	1038	11 825	12 109
	3	4	6,7,16	2,4,10,17	949	2035	942	10 986	10 986
8	1	7	2	4,5,6,10,13,16,17	1708	2096	1380	16 377	18 405
	2	6	2,17	4,5,6,10,13,16	1745	1429	1082	12 062	14 428
	3	5	2,5,13	4,6,10,16,17	974	1544	828	10 643	11 035
	4	4	4,5,10,13	2,6,16,17	1238	1367	797	10 420	10 626
9	1	8	2	5,6,10,13,16,19,21,22	1327	1856	1239	16 174	18 583
	2	7	2,19	5,6,10,13,16,21,22	1075	1251	972	10 906	14 575
	3	6	2,19,22	5,6,10,13,16,21	1264	1170	756	10 812	11 337
	4	5	5,6,13,16	2,10,19,21,22	762	1544	718	10 770	10 770
10	1	9	2	1,5,6,10,13,16,20,21,22	1327	1668	1125	16 340	18 750
	2	8	6,16	1,2,5,10,13,20,21,22	907	1353	884	12 636	14 741
	3	7	5,6,16	1,2,10,13,20,21,22	715	1309	690	11 310	11 502
	4	6	5,6,13,16	1,2,10,20,21,22	749	1314	653	10 883	10 883
	5	5	5,6,10,13,16	1,2,20,21,22	1039	984	607	10 116	10 116

$$\frac{1}{s}\left(\sum_{k=1}^{n} p_k^1 q_k - S^*\right) \geqslant \lambda \frac{\overline{\Pi}^a}{r+s} \qquad (15)$$

$$d_l \in \{0, 1\},\ p_k^1 \geqslant 0,\ p_k^2 \geqslant 0,\ k \in K,\ l \in \overline{X} \qquad (16)$$

Observe that $(P^a_\lambda(\overline{X}))$ reduces to $(P^a(\overline{X}))$ for $\lambda = 0$.

In order to select a location equilibrium, a sequence of problems $(P^a_\lambda(\overline{X}))$ can be solved for fixed increasing λ values until one not feasible problem is found. If $\overline{\lambda}$ is the greater value of λ for which $(P^a_\lambda(\overline{X}))$ is feasible, firms could select the location equilibrium given by the following partition:

$$\overline{X}^1 = \{l \in \overline{X} : \overline{d}_l = 1\}$$

$$\overline{X}^2 = \{l \in \overline{X} : \overline{d}_l = 0\}$$

where $\bar{d}_l$ are the optimal values for variables d_l in problem $(P^a_{\bar{\lambda}}(\overline{X}))$.

5.2. An illustrative example

We present an example with real data in order to illustrate social cost minimization and how to select a location equilibrium when multiple location equilibria exist. Both market areas and location candidates are the Spanish cities on the Iberian Peninsula with a population of over 4000 people (ie 1046 cities). As demand, q_k, a proportion of the total population of city k was taken. Delivered price p_{ik} was taken proportional to the Euclidean distance between cities i and k. The geographical coordinates and population of each city (from the 2001 census) were obtained from http://www.terra.es/personal/gps.2000 and http://www.ine.es, respectively. The social cost minimization problem ($(r + s)$ $-MEDIAN$ problem) was solved for $r + s = 2, 3, \ldots, 10$.

Table 5 Results for $\lambda = 0.7$

$r + s$	r	s	$\overline{X}^1$	$\overline{X}^2$	$\frac{\Pi^1}{r}$	$\frac{\Pi^2}{s}$	$\lambda \frac{\overline{\Pi}^a}{r+s}$	$\overline{\Pi}^a_\lambda$	$\overline{\Pi}^a$
2	1	1					4322		12 348
3	1	2					3227		13 831
4	1	3					2446		13 975
	2	2	2,18	11,15	2101	3136	1833	10 475	10 475
5	1	4					2114		15 097
	2	3	2,18	8,10,15	2003	2423	1578	11 274	11 274
6	1	5					1925		16 496
	2	4	2,19	4,7,10,14	1785	2003	1427	11 583	12 232
	3	3	4,7,10	2,14,19	1644	1714	1175	10 075	10 075
7	1	6	2	4,6,7,10,16,17	1708	2405	1682	16 139	16 821
	2	5	2,17	4,6,7,10,16	1745	1667	1211	11 825	12 109
	3	4	4,7,10	2,6,16,17	1571	1367	1099	10 182	10 986
8	1	7	2	4,5,6,10,13,16,17	1708	2096	1610	16 377	18 405
	2	6	2,17	4,5,6,10,13,16	1745	1429	1262	12 062	14 428
	3	5	2,5,13	4,6,10,16,17	974	1544	966	10 643	11 035
	4	4	4,5,10,13	2,6,16,17	1238	1367	930	10 420	10 626
9	1	8					1445		18 583
	2	7					1134		14 575
	3	6	2,19,22	5,6,10,13,16,21	1264	1170	882	10 812	11 337
	4	5	2,19,21,22	5,6,10,13,16	1189	1094	838	10 227	10 770
10	1	9	2	1,5,6,10,13,16,20,21,22	1327	1668	1312	16 340	18 750
	2	8					1032		14 741
	3	7	2,5,13	1,6,10,16,20,21,22	837	1152	805	10 574	11 502
	4	6	1,2,20,22	5,6,10,13,16,21	990	1121	762	10 688	10 883
	5	5	5,6,10,13,16	1,2,20,21,22	1039	984	708	10 116	10 116

For the optimal solution $\overline{X}$ of each $(r+s)$−*MEDIAN* problem, we solved both problem $(P^a(\overline{X}))$ and problems $(P^a_\lambda(\overline{X}))$, $\lambda = 0.5, 0.6, 0.7, 0.8, 0.9$, for all combinations of r and s. Optimal solutions to these problems are location equilibria. All these ILP problems were solved by using the optimizer *FICO Xpress-Mosel* (2009).

The different optimal locations obtained when solving all the above mentioned $(r+s)$−*MEDIAN* problems are shown in Table 1. The first column is a label to represent each location. Columns 2 and 3 represent the number of the node and the name of the corresponding city. The fourth column is the number of inhabitants of each city. The geographical situation of the cities is shown in Figure 2.

The results for the $(P^a(\overline{X}))$ problem (location equilibria maximizing aggregated profit) are shown in Table 2, where the social cost $(S(\overline{X}))$, the location equilibrium ($\overline{X}^1$ and $\overline{X}^2$), the average profit per facility each firm obtains ($\frac{\Pi^1(\overline{X}^1,\overline{X}^2)}{r}$ and $\frac{\Pi^2(\overline{X}^1,\overline{X}^2)}{s}$), and the maximum aggregated profit ($\overline{\Pi}^a$), are given for each one of the 25 scenarios considered. It is possible to observe how social cost $S(\overline{X})$ reduces as the number of facilities to locate $(r + s)$ increases. The reduction in social cost per each additional facility is more important when the number of facilities to locate is small. The location equilibrium is quite unstable to changes in the number of facilities to locate. Nevertheless, as the number of facilities increases, there seems to be a stable group of locations coincident in all scenarios. The most frequently appearing nodes are 2, 10, 16, and 6, in this order. Note that differences in average profit per facility between firms are in some cases very important. For example, when $r + s = 8$, average profit per facility of firm 2 is more than seven times the average profit per facility of firm 1 when the difference in the number of facilities is high ($r = 1$, $s = 7$), and more than two times when both firms locate the same number of facilities ($r = s = 4$). For any fixed

Table 6 Results for $\lambda = 0.8$

$r + s$	r	s	$\overline{X}^1$	$\overline{X}^2$	$\frac{\Pi^1}{r}$	$\frac{\Pi^2}{s}$	$\lambda \frac{\overline{\Pi}^a}{r+s}$	$\overline{\Pi}^a_\lambda$	$\overline{\Pi}^a$
2	1	1					4939		12 348
3	1	2					3688		13 831
4	1	3					2795		13 975
	2	2	2,18	11,15	2101	3136	2095	10 475	10 475
5	1	4					2416		15 097
	2	3	2,18	8,10,15	2003	2423	1804	11 274	11 274
6	1	5					2199		16 496
	2	4	2,19	4,7,10,14	1785	2003	1631	11 583	12 232
	3	3	4,7,10	2,14,19	1644	1714	1343	10 075	10 075
7	1	6					1922		16 821
	2	5	2,17	4,6,7,10,16	1745	1667	1384	11 825	12 109
	3	4	4,7,10	2,6,16,17	1571	1367	1256	10 182	10 986
8	1	7					1840		18 405
	2	6					1443		14 428
	3	5	2,5,17	4,6,10,13,16	1273	1237	1103	10 006	11 035
	4	4	4,5,10,13	2,6,16,17	1238	1367	1063	10 420	10 626
9	1	8					1652		18 583
	2	7					1296		14 575
	3	6	2,19,22	5,6,10,13,16,21	1264	1170	1008	10 812	11 337
	4	5	2,19,21,22	5,6,10,13,16	1189	1094	957	10 227	10 770
10	1	9					1500		18 750
	2	8					1179		14 741
	3	7	2,6,16	1,5,10,13,20,21,22	1047	932	920	9667	11 502
	4	6	1,2,20,22	5,6,10,13,16,21	990	1121	871	10 688	10 883
	5	5	5,6,10,13,16	1,2,20,21,22	1039	984	809	10 116	10 116

value of $r + s$, it is observed that average profit per facility of a firm increases when the number of its facilities increases. On the other hand, it is possible to see that the aggregated profit reduces as $s - r$ reduces too. This is motivated by the effect of the increase in competition, and therefore the reduction in individual profits. Observe that different profits are obtained when the firms have the same number of facilities. This is explained by the fact that customers buy from the closest facility, then the markets captured by the firms are different (for instance, see in Table 2 the optimal facility locations obtained for $r = s = 2$), and the firms obtain different profits.

The results for the $(P^a_\lambda(\overline{X}))$ problems are shown in Tables 3 to 7. These tables contain location equilibria maximizing aggregated profit but verifying the equity constraints. The optimal value of $(P^a_\lambda(\overline{X}))$ is denoted by $\overline{\Pi}^a_\lambda$ and it is shown in column 9. These results are summarized in Figure 3, where for each λ value the first bar contains three levels: number of scenarios in which the found equilibrium is different form the one obtained for $\lambda = 0$ (top level); number of scenarios in which the found equilibrium is the same as the one obtained for $\lambda = 0$ (mid level); and number of scenarios in which the equilibrium is lost (bottom level). The second and third bars are the maximum percentage deviation and the average percentage deviation of Π^a_λ with respect to $\overline{\Pi}^a$, respectively. The percentage deviation is measured by $100(\overline{\Pi}^a - \overline{\Pi}^a_\lambda)/\overline{\Pi}^a$. We see that for some scenarios, as λ increases, it is not possible to find any equilibrium that satisfies the equity constraints. Furthermore, new location equilibria for other scenarios may appear when equity constraints are considered. Thus, for $\lambda = 0.5$ there is one scenario without location equilibrium and 12 scenarios with a new location equilibrium, and for $\lambda = 0.9$ there are six scenarios where the location equilibrium is lost in which

Table 7 Results for $\lambda = 0.9$

$r+s$	r	s	$\overline{X}^1$	$\overline{X}^2$	$\frac{\Pi^1}{r}$	$\frac{\Pi^2}{s}$	$\lambda\frac{\overline{\Pi}^a}{r+s}$	$\overline{\Pi}^a_\lambda$	$\overline{\Pi}^a$
2	1	1					5557		12 348
3	1	2					4149		13 831
4	1	3					3144		13 975
	2	2					2357		10 475
5	1	4					2717		15 097
	2	3					2029		11 274
6	1	5					2474		16 496
	2	4					1835		12 232
	3	3	4,7,10	2,14,19	1644	1714	1511	10 075	10 075
7	1	6					2163		16 821
	2	5	2,17	4,6,7,10,16	1745	1667	1557	11 825	12 109
	3	4					1412		10 986
8	1	7					2071		18 405
	2	6					1623		14 428
	3	5					1241		11 035
	4	4	4,5,10,13	2,6,16,17	1238	1367	1195	10 420	10 626
9	1	8					1858		18 583
	2	7					1457		14 575
	3	6	2,19,22	5,6,10,13,16,21	1264	1170	1134	10 812	11 337
	4	5	2,19,21,22	5,6,10,13,16	1189	1094	1077	10 227	10 770
10	1	9					1687		18 750
	2	8					1327		14 741
	3	7					1035		11 502
	4	6	1,2,20,22	5,6,10,13,16,21	990	1121	979	10 688	10 883
	5	5	1,2,20,21,22	5,6,10,13,16	984	1039	910	10 116	10 116

there was a location equilibrium for $\lambda = 0.8$. For $\lambda = 0.9$ there is a location equilibrium in seven out of 25 scenarios. Observe that the width of the bottom level increases while the width of the mid level decreases when λ increases. The maximum percentage deviation of $\overline{\Pi}^a_\lambda$ with respect to $\overline{\Pi}^a$ is 25, which is obtained for $\lambda = 0.8$. The average percentage deviation increases when λ increases and it varies from 5.9 to 20.4.

For each λ value, we have also evaluated the percentage deviation of the average profit per facility of each firm with respect to the global average profit per facility which are given by $100\frac{(\Pi^1/r-\overline{\Pi}^a/r+s)}{\overline{\Pi}^a/r+s}$ and $100\frac{(\Pi^2/s-\overline{\Pi}^a/r+s)}{\overline{\Pi}^a/r+s}$, respectively. Thus, the percentage deviations obtained from Table 2, corresponding to $r + s = 4$ are $-55{,}3$ ($r = 1$) and 19,7 ($r = 2$) for firm 1 and 18,4 ($s = 3$) and $-19{,}8$ ($s = 2$) for firm 2, then the average deviations are $-17{,}8$ for firm 1 and $-0{,}7$ for firm 2. The average deviations corresponding to the

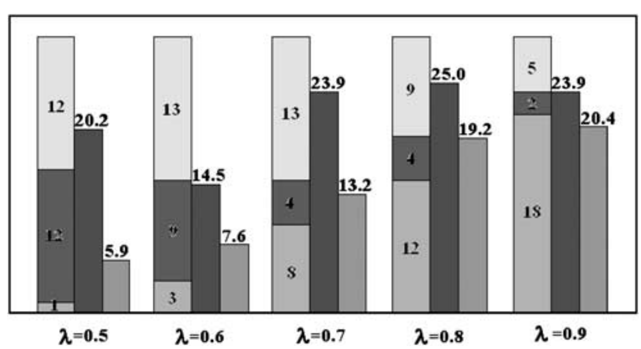

Figure 3 Results for the $(P^a_\lambda(\overline{X}))$ problems.

two firms for each $r + s$ value, without equity constraints ($\lambda = 0$) and with equity constraints ($\lambda = 0.5, \ldots, 0.9$), are shown in Figure 4, where it is observed that deviations above

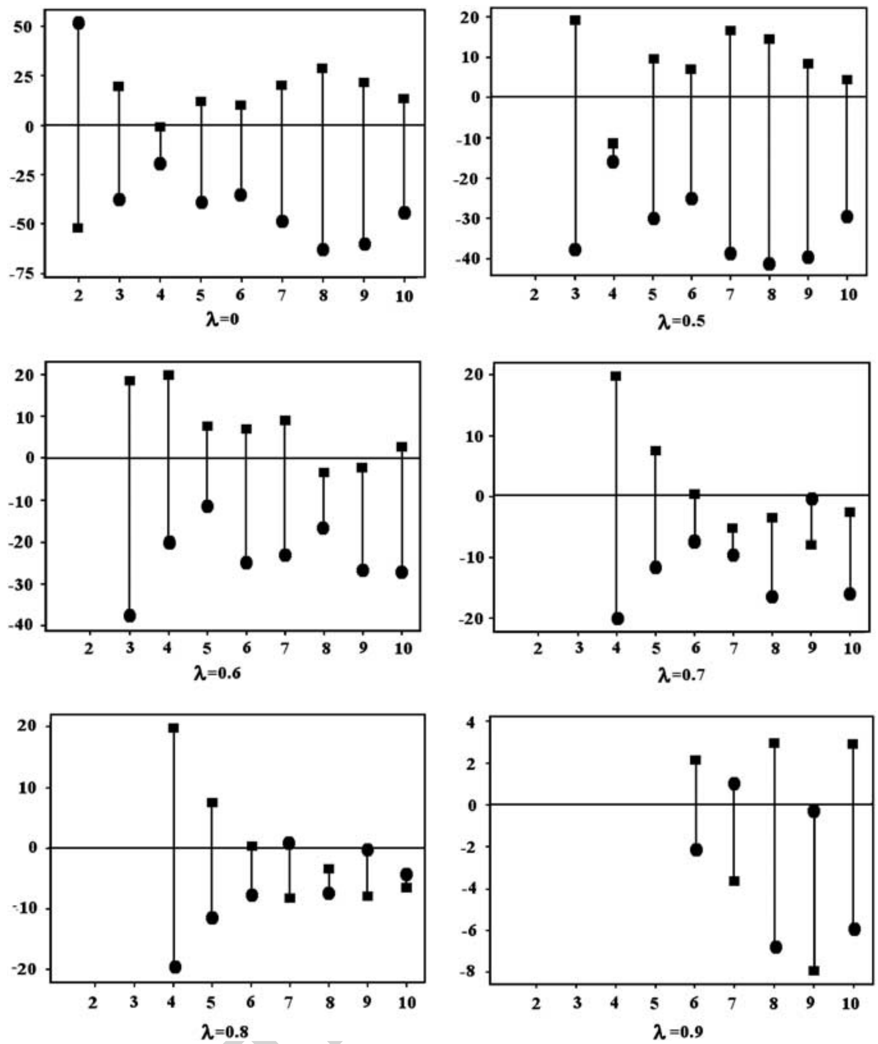

Figure 4 Average deviations for each $r + s$ value and each λ value.

the global average profit per facility are smaller than deviations below the global average profit per facility. Normally, one firm deviates above and another deviates below, but in some cases both firms deviate below the global average profit per facility (for instance, $\lambda = 0.7$ and $r + s = 7$, 8,10). Observe that deviations decrease when λ increases. The maximum deviation for $\lambda = 0.9$ varies form 2% ($r + s = 6$) to 8% ($r + s = 9$).

Finally, as a way of evaluating the loss of efficiency (reduction in aggregated profit) in the proposed solution when introducing equity constraints, we compare Tables 2 and 7 ($\lambda = 0$ and $\lambda = 0.9$, respectively) with respect to the seven scenarios with equilibrium. It can be observed how in two scenarios the solution proposed in each one of them is the same in both tables, and therefore efficiency is not lost ($r = s = 3$ and $r = s = 5$); in three scenarios the reduction in aggregated profit is close to 2% ($r = 2, s = 5$; $r = s = 4$; $r = 4, s = 6$); and in the other two scenarios this reduction is close to 5% ($r = 3, s = 6$; $r = 4, s = 5$).

6. Extensions to oligopoly and conclusions

The previous results for two firms can directly be extended to oligopoly. In fact, long-term price competition for a fixed number U of firms, which locate several facilities each, leads to a price equilibrium in which each firm sets the lowest delivered cost of its competitors as the price in market k, if that firm is the only one with minimum delivered cost in market k. Otherwise, the firm sets its minimum delivered cost as the price in market k. In a similar way to the duopoly case, equilibrium prices are determined for the chosen facility locations. For any location space, it is possible to see that the locations of the firms are in equilibrium if each firm minimizes the social cost with respect to the competitors' fixed locations. In the case of the location space being a network, a location equilibrium also exists at the nodes.

As in the duopoly case, a location equilibrium can be obtained by minimizing social cost. This problem can be formulated in discrete space as an ILP problem by taking variables x_i^u and z_{ik}^u, $u = 1, \ldots, U$, which in turn becomes a $(\sum_{u=1}^{U} r^u) - MEDIAN$ problem if location candidates and delivered prices are the same for all competing firms, where r^u is the number of facilities to be located for firm u. Furthermore, ILP formulations similar to the ones in the duopoly case, can be used to select a location equilibrium when equity constraints are considered.

We have presented a general framework of the location-price Bertrand game where minimizers of the social cost are location equilibria, and shown that a location equilibrium in discrete location space can be found by solving an ILP problem. A case where multiple location equilibria exist has been considered, for which ILP formulations, without and with equity constraints that depends on a parameter λ, $0 \leqslant \lambda \leqslant 1$, have been proposed to select one location equilibrium. The higher the value of λ the more equity in profit per facility for each competing firms is obtained, however a reduction in aggregated profit may occur. An example with real data has been solved in 25 scenarios to point out the existence of multiple equilibria and how the equity constraints affect them. For the highest value of λ studied ($\lambda = 0.9$), there exist a location equilibrium in more than 25% of the scenarios considered. In all the scenarios with equilibrium for $\lambda = 0.9$, the *loss of efficiency* (reduction in aggregated profit) in less than 5%, and in 5 of these 7 scenarios the loss of efficiency is less than 2%.

In a network we have shown that, under two common assumptions, the problem of determining a location equilibrium is reduced to a discrete problem in which the location candidates are the nodes. Then the proposed procedures can be used when the location space is a network.

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